RECAP

THE
UNIVERSITY OF MISSOURI
STUDIES

EDITED BY
W. G. BROWN
Professor of Chemistry

AN INTRODUCTION TO THE
MECHANICS OF THE INNER EAR

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MAX MEYER, Ph. D.
Professor of Experimental Psychology

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PREFACE

About two thirds of this study has been published at different times in various German scientific periodicals, chiefly in the Zeit-schrift für Psychologie und Physiologie der Sinnesorgane. The author has long hesitated to present in book form the results of his labor in this remote corner of scientific investigation because the interest in these problems seems to be neither intense nor general. This lack of interest on the part of the scientific public, however, is not due to the unimportance of the subject, but rather to a wide-spread conviction that all the problems pertaining to it were solved half a century ago and that therefore nothing problematic is left. For years during which — since his student days — these problems have been in the mind of the writer, he has belonged to an exceedingly small minority of scientific men, who have not permitted themselves to become captives of this conviction. But since this minority is gradually increasing in number, and since professional friends have encouraged the writer he has decided to lay before the public the results of his investigations in a continuous exposition of his theory as far as it goes at present. It is natural that he has preferred to do this in the English language, since nearly all his previous publications concerning it are in German.

The author does not pretend to present in this book a complete, perfect, and final solution of the problem concerning the mechanics of the inner ear. His farthest reaching hopes will be fulfilled if he succeeds in impressing upon the reader's mind the fact that there are here still problems left for solution and in giving these problems such a clear and definite formulation that the interest of others will be turned towards them. There is little hope for a final solution of these problems except by the co-operation of many investi-

gators. The contents of this book are arranged from a peda-
gogical rather than from a logical point of view. The author
does not intend to present a systematic representation of his
own ideas for comparison with the ideas of others, but rather
a series of lectures as he would deliver them before a class of
college students, not presupposing any knowledge or any in-
terest but what a somewhat advanced college student might
be expected to possess. A reader who should prefer to make
himself acquainted with the contents of this book from an-
other point of view, will be able to do this by the aid of the
index added.

The author has attempted to omit as much as possible
everything of a polemic nature. His criticism of the views
of other investigators may be found in his previous publica-
tions. In this book he does not propose to record the views
of other scientists, but the conclusions which he has reached
himself after more than a decade of thought concerning these
problems. For the reader who might be interested in the
development of the author's thought concerning these prob-
lems, he has added at the end of the book a list of those pub-
lications of his own which are directly concerned with the
problems here presented.

The author hopes that this booklet will help to break
down the barrier of dogmatism which has too long stood in
the way of progress in this field of scientific inquiry, and
which is still far from being a thing of the past. It is truly
dogmatism to profess that the application of so simple a
theorem as that of Fourier can do justice to an attempt at
comprehending the mechanical processes underlying the won-
derfully complicated and unfortunately only superficially
known phenomena of audition.

THE MECHANICS OF THE INNER EAR

Everyone knows that the part of our body which in ordinary life we call the ear and which anatomists call the pinna, is not the organ of hearing but a mere appendage to the organ. Its chief utility consists in the fact that it aids us in distinguishing sounds coming from a source in front of us from sounds in our rear. We know how much more difficult it is to understand the words of a speaker behind us than the words of one who stands before us. We can reverse this condition by forming of our hands leaves similar to the external ears, but naturally larger and placing them opposite the ears, that is in front of the opening, the auditory passage. Then, sounds from the rear can enter the passage and reach the tympanum with a much greater force than sounds coming from the front. Animals, being able to move their external ears, can use them, of course, to greater advantage than human beings.

The external ear

The organ of hearing—in the narrower sense of the word—that is, the anatomical structure within which the ends of the auditory nerve fibres receive their peripheral excitations, is to be found stretched out along the central line of a tube which is very narrow relative to its length. This tube is called by the anatomists the cochlea, because it is not built in the form of a straight line, but coiled up like the tube of a snail shell. The advantage of its being coiled up in this way is obviously not to be sought in its mechanic—or rather hydrodynamic—function. At least, no

The tube containing the sense organ is long and narrow

one, to the writer's knowledge, has ever expressed himself as inclined to look for it there. For its hydrodynamic function it is clearly of no great importance whether the tube is curved or straight, and we shall speak of it in the following for the most part as if it were straight, in order to simplify the discussion. The real advantage of this shape of the tube is doubtless a mere anatomical one, it being possible thus to find a better place for it in the base of the skull.

We must, in order to understand the function of this tube, be aware of the fact that it is filled with a watery fluid, lymph, and that its walls consist of hard unyielding bone. Now, when we go **The contents of** through the literature of the subject, we **the tube, a fluid,** often see writers speak of waves in the fluid **is incompressible** which are said to pass along the tube as air waves move in a tube filled with air. Views of this kind cannot, of course, contribute towards an understanding of the process of stimulation of the peripheral nerve ends. They are not rational considerations of the facts before us, but theoretical dreams, forgetting the physical conditions of the case. Let us regard the velocity of the sound in such a fluid as that of the inner ear as about fourteen hundred meters, let us remember that the whole length of the tube is only a couple of centimeters, let us understand, then, that even with rather high tones of short wave lengths— beyond the musical range—the total length of the tube is only a small part of the spatial length of the waves said to travel up and down the tube; and we shall admit at once that to speak of tone waves travelling in the lymph up and down the tube is like speaking of a horse race which is to take place within a dog kennel. We have to follow the custom of the physicists who in such cases neglect the compressibility and elasticity of the small volume of fluid altogether. We must, therefore, regard the fluid in the cochlea as being of identical

density throughout at any given time, that is practically, as unelastic, incompressible.
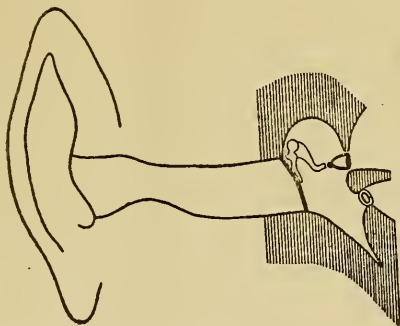

Fig. 1.   The external and the middle ear

The walls of the tube consist of hard, unyielding bone, except in two places where the bone is broken through and the openings closed by flexible membranes. These two places are commonly called the oval and the round windows. (The fact that the tube communicates with the semicircular canals and the other parts of the labyrinth can here be neglected, since all these communicating cavities are also enclosed in bone, not possessing any windows.) On the other side of these windows there is the air of the middle ear. Let us now consider at once what could happen to the fluid in the tube if rhythmical changes of pressure in the external air (a "tone") caused, through the tympanum, like changes (of condensation and rarefaction) in the air of the middle ear. Let us at present, however, consider this under the imaginary assumption of no chain of ossicles existing in the middle ear. What was said about waves in the fluid of the tube holds good to some extent also for the air in the middle ear. That which occurs there is the same as that which occurs, say, in a bicycle

The tube has two windows to communicate with the middle ear

pump, that is, an alternate condensation and rarefaction of all the particles of air almost simultaneously. This condensation and rarefaction always acts in the same sense (positive or negative) on both windows of the tube. According to the laws of hydrodynamics no motion in the fluid of the tube can result from the difference in size of the two windows. It is hardly comprehensible, therefore, why we find in literature lengthy discussions of the question whether it is the round or the oval window through which "the tone waves" enter the inner ear. They do not enter through either window since they do not occur in the middle ear, the volume of this cavity being too small to contain whole tone waves. Only after complete destruction of the tympanum would the question as to the manner in which an air wave strikes the two windows attain practical importance. Under normal conditions we must regard all the air particles in the middle ear as being, at any time, of identical density, and, thus, as unable to produce any movement in the inner ear.

If there were no ossicles, the fluid in the tube would remain practically motionless. But to the membrane of the oval window is attached the plate of the stirrup which has a somewhat rigid connection with the tympanum. The result is that every movement of the tympanum is accompanied by a movement of the stirrup in the same (positive or negative) direction. Whenever the tympanum moves inwards, the air in the middle ear is, of course, somewhat condensed. But this condensation or rarefaction has no relevant influence on the fluid in the tube, as before mentioned. The alternate condensation and rarefaction of the air in the middle ear, resulting from like processes in the external auditory passage, is an unavoidable, but functionally negligible by-product of the mechanical process in question, bearing no direct rela-

Disturbances within the tube are caused by motion of the stirrup

tion to the function of the tube. It is the movement of the stirrup which causes the disturbances in the fluid of the tube which we have soon to study in detail. And this motion of the stirrup is made possible only through the mediation of solid bodies, the auditory ossicles.

The bony connection between the stirrup and the tympanum would serve its purpose of causing movements in

The auditory ossicles are a system of levers

the fluid of the tube whatever might be the special structure of this connecting link. As a matter of fact, it is arranged in such a particular manner that it acts as a lever (or system of levers), the large arm, so to speak, being attached to the tympanum, the small arm to the stirrup. This effect, however, is produced in different animals in different ways. In birds, for example, (Fig. 2) there is no chain of three little bones, but only a single bone, a rod bearing an oval plate. The leverage of this simple connection is explained by the fact that the tympanum and the window plate are not in parallel planes. The far

Fig. 2. Schematic representation of the leverage in birds

more complicated connection by means of three links of a chain of bones in most of the mammals has been theoretically studied by various investigators and found to result in a similar, but probably more delicately adjustable leverage than the simpler arrangement in birds. The advantage of the lev-

erage is easily understood. To cause a fluid to move along
a narrow tube requires a considerable force because of the
friction   resulting from the   narrowness   of   the   passage.
The   extent   of   movement,   on   the   other hand, may be
of any minuteness, the nerve ends certainly being sensitive
to the very slightest curving of their tufts of hairs of which
we shall have to speak again. It is of advantage, therefore, to
gain force at the expense of magnitude of displacement.

Someone might here raise the question : Why are there
two windows when only one of them has a solid connection

with the tympanum? The answer to this
question is very simple. If there were
not a second window, the stirrup could
not move at all. Imagine a bottle filled
with water up to the stopper and the

**Why would not
one window
be sufficient?**

stopper fitting the neck most accurately. Would it be possi-
ble to drive the stopper farther in? The water being incom-
pressible, it would not be possible for a moderate force
to drive a perfectly fitting stopper in any more than to pull it
out.   The second window, closed by a flexible membrane,
is therefore necessary if the movements of the stirrup and
of the fluid in the tube are to take place. If it were not for
movements of the fluid, the round window would be super-
fluous. It is, however, not an essential condition that the sec-
ond window should open on the middle ear and not perhaps
directly on the external air space; for instance, on the exter-
nal auditory passage, or anywhere on the skull. But it is
an essential condition that the one window containing
the stirrup plate open on a drum and that the plate be
rigidly connected with the external membrane of this
drum. Thus every condensation or rarefaction of the
air outside the drum must result through movements of the
tympanum in like condensations or rarefactions inside the
drum ; the movements of the tympanum must result in move-

ments of the stirrup, and consequently in movements of the fluid in the tube. If the tympanum is destroyed to such an extent that the middle ear can no longer act even imperfectly as a drum, movements of the fluid in the tube must be difficult to produce. The organ is then deprived of its normal manner of functioning—a defect which does not necessarily involve total deafness, yet certainly a great impairment of the sense of hearing.

We naturally do not wonder at the fact that the round window is arranged in the simplest way possible, that is, opening on the middle ear not far from the oval window.

Let us now attempt to determine what movements would occur in the tube, caused by movements of the stirrup, if this tube were a perfectly plain tube, containing nothing whatever but an incompressible fluid. It is a decided advantage to study first a case as simple as can be imagined. We are sure that, thus, the elementary foundations of our thought will be clear and not confused by the influence of a complexity of conditions and a sum of powerful prejudices which almost inevitably ac-
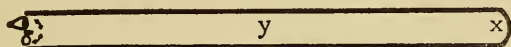
The movement of the fluid in a plain tube

Fig. 3. Movement of fluid in a plain tube

company a complexity of conditions. Let us try to keep clear of such influences. In figure 3 we see the anatomical facts of our imaginary case diagrammatically represented: a long and narrow tube, two windows at one end, one of these windows containing the stirrup, the other end of the tube closed.

The question is this: What will happen to the particles of fluid in the tube when the stirrup moves slightly inwards or outwards? This is a problem which can be answered either on the basis of our general knowledge of similar processes or by means of a special experiment. Let us first try the former way. When the stirrup is pushed inwards and the round window outwards, the liquid near the windows must certainly move in the direction indicated by the arrows in the figure. Of course, the direction of the movement would be the opposite if the movement of the stirrup changes its sign and pulls instead of pushes. But what would happen in the fluid at the other end of the tube? At **x** or even at **y**? The answer to the question is simple: Nothing would happen. No movement of any kind could possibly occur there, since there is no sufficient cause why any movement should occur. The friction of the fluid against the walls of the tube, which is quite considerable in a narrow tube, must prevent any spreading of the disturbance beyond a very near limit. That is, whenever the stirrup moves back and forth, those particles of the fluid which are in the nearest path leading from the oval to the round window must move accordingly. All the rest of the fluid remains motionless.

In order to demonstrate the facts just mentioned to those finding difficulty in understanding that from the general laws of hydromechanics nothing else could re-
A simple sult in the case in question but what we
experiment have just described, we may perform the following experiment. A box containing white clay in a plastic condition has two circular openings on one side, not far from each other, as shown by figure 4 in cross-section. We now press, by means of a piston, into one of the openings, A, a small quantity of colored clay, then a small quantity of white clay, and again colored clay until the latter becomes visible on the outside of the box

at the other opening, B. In our figure we see at **a** and **b**
the colored clay pressed in first. The part protruding be-
yond the outside of the box is cut away. At **c** we see the
white clay pressed in afterwards, and at **f** the advance guard,
so to speak, of the colored clay pressed in last. What has
happened within the box is obviously this. The colored clay
pressed in first, collects inside the box near A in the direc-
tion of B. A corresponding amount of the white clay with
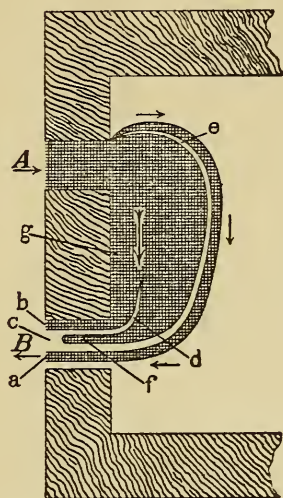
Fig. 4.   An experiment with
plastic clay

which the box was filled has been pushed out through the
opening B. The white clay pressed in next forces up the
colored clay somewhat as a mass of glass is blown up in a
glass factory to form a bottle. This white clay is forced up in
turn by the succeeding colored clay, the "bottle" of colored
clay increasing its dimensions at the same time. During this
whole time and afterwards the total mass moves in the di-
rection of B. However, the particles of clay to the left,

nearer the openings, move much more quickly than those
farther to the right. This is seen from the fact that the
left wall d of the white "bottle" has been separated entirely
from the opening A and is just getting ready to disappear
altogether through the opening B, whereas the right wall e
is merely beginning to sever its connection with A. We
have here a simple experimental proof for the statement of
the preceding paragraph that friction prevents the spreading
of the motion beyond narrow limits, causing it to occur as
near the two openings as possible. Although the experiment
in this form does not show it, the reader hardly doubts that
somewhat farther to the right, say six inches from the open-
ings, no motion whatsoever has occurred during the whole
time. The quickest motion, of course, is in this particular
case not found at the extreme left, at g, but about a fourth
of an inch to the right, since the friction at g is too great.
Without entering into a detailed study of the hydrodynamic
problem which confronts us here, in which friction against
the walls, internal friction in the fluid, and the momentum
of the fluid play their roles, let it be sufficient to say here
that the motion is practically limited to the portion of the
tube near the windows in accordance with the general law
of nature that whatever occurs, occurs with the least pos-
sible expenditure of energy. Some clay is pressed in at A.
The same quantity has to pass out at B. This can be made
possible by many kinds of displacement of the particles with-
in the box. But only one form of displacement becomes
actual, the one that requires the smallest amount of work to
be done by the piston at A. And this form of displacement
consists in the displacement being confined to the neighbor-
hood of the openings.

Let us now consider another imaginary case which will contribute towards a better understanding of the processes

The effect of a rigid partition within the tube

actually occurring in the ear. Suppose a part of the tube, near the windows, to be divided by an inflexible partition, as shown in figure 5. It is self-evident that in this case every movement of the stirrup would cause the particles of fluid in the upper and lower division of the tube to move in the directions of the arrows, parallel to the partition; and the particles at y, at the end of the partition, to move up or down. But the fluid farther on in the undivided tube would remain motionless, as in the former case, since there is no sufficient cause why it should move. If the partition extended farther, the only change re-
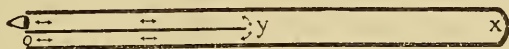
Fig. 5. A rigid partition in the tube

sulting would be a diminution of the length of that part of the tube where the fluid remains permanently motionless. If the partition extended to x (Fig. 5), leaving only a small opening of communication between the upper and lower division, all the fluid within the tube would have to move whenever the stirrup moves. If the partition extended throughout the tube, leaving no communication whatever between the two divisions, no movement of the fluid could then take place, of course; but no piston-like movement of the stirrup could then take place either.

Let us now imagine a third case. Suppose a partition
to divide the tube lengthwise into two divisions, leaving,
however, a small opening of communica-
tion between the divisions at x. Suppose
further this partition to be neither per-
fectly rigid like a wall of hard bone nor
as readily yielding and in turn contract-
ing as a thin rubber membrane, but to be
of the physical nature of a soft leather strap somewhat
loosely stretched out between the opposite sides of the
tube to which it is assumed to be well attached. To have
something definite in mind, let the reader think, for compar-
ison of its function, of a leather-seated chair. If you press
from below, the seat yields and bulges upwards; but
soon it stops in spite of your effort. If now you sit down
on the chair, the seat bulges downwards; but again, it
soon stops—how could it otherwise be used for the support
of your weight? But what is particularly important to note
here, is the fact that the leather seat, after it has bulged
either way, may continue to remain thus until some ex-
ternal force acts upon it again from the other side. Now
let us consider the movements which would occur in the
fluid of a tube, divided into two divisions by a partition
of the nature just described. If the partition could yield
indefinitely, the case would obviously be practically the same
as the first one we studied—without any partition. That
is, the fluid would move near the two windows and the part
of the partition suspended between moving masses of fluid
would move with the fluid. Farther on where the fluid re-
mains motionless the partition would remain motionless too.
But we made the assumption that the partition, like the seat
of a leather-seated chair, can move only within certain
narrow limits up and down. Now, the result of this condi-
tion will be this. When the stirrup begins moving inwards,

The effect of a
flexible, but inelas-
tic partition with-
in the tube

the part of the partition next to the windows must follow
the movement of the fluid and move downwards.   But soon
it has reached its lower limit.   Consequently it acts now as
an unyielding partition, the effect of which we studied in
our second case above.   The fluid just above and below this
temporarily unyielding part can now move only horizontally,
but the particles of fluid next to the end of this now motionless
piece move down and push the underlying piece of the par-
tition down until it has reached its lower limit.   And so,
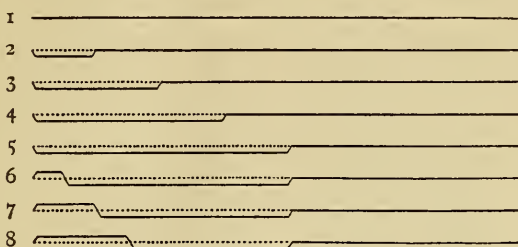gradually, further and further pieces of the partition come

Fig. 6.   The partition moves within an upper
and a lower limit

down until the stirrup stops moving inwards.   Figure 6
shows a number of successive stages of the position of the
partition during this process.   The vertical scale in this rep-
resentation is, of course, enormously exaggerated relative to
the horizontal scale.   But at once after stopping, the stirrup
begins to move in the opposite direction.   At once the par-
ticles of fluid next to the windows (not those which have
moved down last) move upwards and take the corresponding
part of the partition with them until it has reached its upper
limit.   Now the following parts come up, and so on in exactly
the same way as before, except that we have now an upward
instead of a downward movement, — until the stirrup
stops moving in this direction.   Let us remember by all

means, because a mistake made here in our comprehension of the process would result in serious errors later, that the bulging of the partition, whether up or down, begins inevitably as near the two windows as possible, and that further pieces can bulge in either direction only under the condition that all the pieces nearer the windows have already reached their limit in that same direction.

We made at the beginning of the last paragraph the assumption that there was a small opening between the two divisions at the extreme end of the tube. A safety valve Let us see what purpose such an opening could serve. What would be the result of an extraordinarily large movement of the stirrup, so large that the whole length of the partition would reach its—upper or lower—limit of position before the stirrup ceased to move in the same direction? The result would be either an enforced stop of the movement of the stirrup or, if the external force acting on the tympanum and stirrup was too violent, a bursting of the partition. The latter disastrous result, however, can to a considerable extent be guarded against by the opening in question. As soon as the total length of the partition has bulged the fluid will begin to flow through this opening from one division of the tube into the other, until the stirrup stops moving in the same direction. Such an opening therefore can serve as a kind of safety valve for the protection of the partition.

After having studied the hydromechanical function of several imaginary tubes with divers interior equipments, let us now turn to a careful survey of the facts which the anatomists have discovered for us concerning the structure of the inner ear. Figure 7 shows us in a cross-section all the important details which have been found there by the anatomists. Hard bone pro-
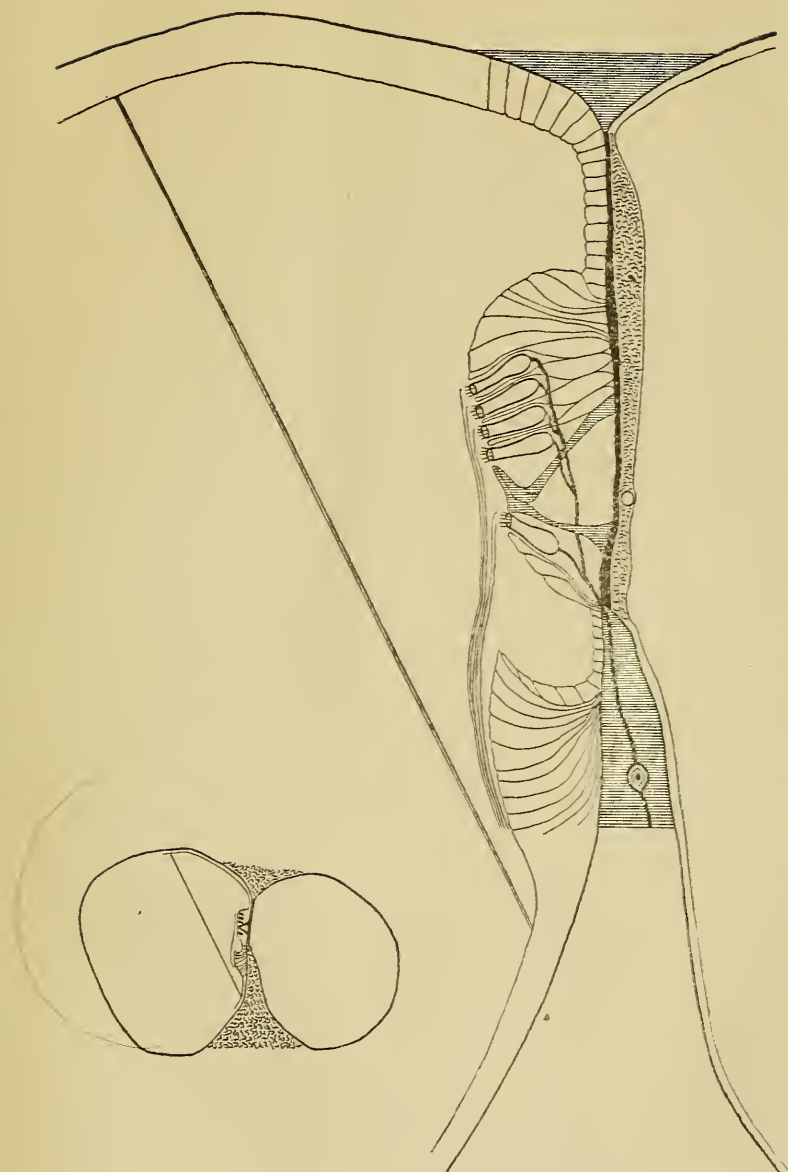
Fig. 7. To the left above is shown a section through the tube, with its two divisions and the partition. Below is shown a more magnified section of the partition alone

trudes from diametrically opposite sides of the bony wall of the tube, on the left side more than on the right.   But the bone does not protrude far enough to actually cut off the lower part of the tube from the upper.   While, therefore, we do not find a hard, inflexible partition, we find indeed some kind of a partition since the space between the bony protrusions is filled with a delicate structure which we shall have to study somewhat in detail.   This structure, which we shall   always   refer   to   hereafter   as   "the   partition"   in   the inner ear, is customarily spoken of under the name of its discoverer as the organ of Corti.   The lower part of this partition has been shown to be a membrane, generally called the basilar membrane.   This is obviously the strongest part of the partition, capable more than any of the other elements of structure to resist a pressure of the fluid above or below.   But we must not think that this membrane is the main part of the partition considering its volume.   It is rather small in bulk compared with the rest.   Above the membrane we see a triangular structure, something like two pillars which have fallen towards each other.   This structure is usually called the rods of Corti. Its mechanical significance becomes at once clear to us when we see at its sides the delicate end organs of the auditory nerve fibres. These end organs would obviously be crushed by the push of the fluid which occurs now from above, now from below, as we have seen, if they were not braced by this arch.   No better protection could be devised than this triangular structure which effectually preserves the natural form of the soft tissues as the skeleton does in the total animal body, without interfering with a slight bending or compression of the tissues of the partition. On the upper side of the partition, opposite the basilar membrane, we notice another membrane, but much more delicate in structure, easily torn to pieces when sections are made for the miscroscope.   This membrane touches the tufts of hairs

which are the extreme peripheral parts of the sensory organs. This membrane, however, is firmly attached to the left side of the partition only. Its right end is free or seems to be almost free. The kind of action exerted by this membrane upon the hair tufts can only be guessed. The real connections between, and the physical properties of, these tissues are not well enough known. We may perhaps make this action a little clearer by assuming that the upper membrane, when the partition bulges upwards, pulls the hairs slightly, and that a bulging of the partition downwards means merely a relief from this pull. It is hardly worth while, however, to enter into details of a function which cannot be more than hypothetical since there are no data upon which to base any more definite theory. But there is little doubt, that the points between the tufts of hairs and the membrane in question are to be regarded as in the strictest sense the periphery of the sensory apparatus of hearing. And we shall scarcely make a grave mistake in assuming that a double bulging, back and forth in the vertical direction, of the partition causes a single shock in all those nerve fibres whose termini are located in this part of the partition, and that somewhere in the neurons a new process, perhaps a kind of chemical process, is set up if more than one of such shocks are received in quick succession, that the special character of this new process is dependent on the frequency with which these shocks follow each other, and that thus we perceive a definite tone, occupying—according to the frequency of shocks received—a definite point in the total series of sensations of hearing.

In the preceding paragraph we studied briefly the anatomical elements of the partition in their mutual relations. We now have to get a definite idea of the physical properties of the partition as a whole in its relation to the surrounding fluid. These properties depend, of course, on the properties of its elements. The partition as a whole can certainly not be regarded as perfectly rigid and unyielding to pressure. It consists of tissues too soft to be unyielding. On the other hand, we cannot possibly assume that under the influence of pressure the partition could bulge to any large extent, for this would be disastrous to the delicate end organs of the nerve fibres. We could hardly make a mistake, then, in assuming that the partition can yield, but only withm very narrow limits up as well as down, even if we did not know anything about the physical properties of the anatomical elements. We know, however, that the basilar membrane is a comparatively tough structure, probably capable of considerable resistance. We are justified, then, in our conviction that the whole partition bulges in response to pressure but resists such pressure as soon as a certain rather narrow limit of displacement is reached.

The physical properties of the partition as a whole

Here, however, arises another question of the greatest importance, which, unfortunately, cannot be answered with anything approaching accuracy. This is the question as to the elasticity of the partition. Of course, all the elasticity the partition can possibly have must be the elasticity of the basilar membrane. The basilar membrane is the only one of the anatomical elements of the partition which might have a tendency to restore spontaneously the whole partition to its original position after the pressure causing the displacement has ceased and before any pressure in the opposite direction has had time to act towards this result.

There is only one way of deciding for our present pur-
pose the question as to the elasticity of the basilar mem-
brane.   We must recall our knowledge
Is the basilar   of the elastic properties of similar mem-
membrane elastic? branous tissues which are found in divers
parts of the human body and elsewhere
in the organic world. Now, we know that there are plenty
of   membranes in the body which, when stretched within
certain limits, show a tendency to return to the original
shape.   But they never remain in a stretched condition, that
is, under tension, for any length of time. Indeed, they would
become permanently lengthened if they remained thus.   This
is the consequence of a universal biological law.   We may,
for instance, bend a sapling and expect it to straighten itself
as soon as we let it go, because of the elasticity of the
stretched tissues of the convex side and the compressed tis-
sues of the concave side.   But if we tie it in this bent po-
sition to 'another tree and return after a year and cut
the tie, we find that it has adjusted itself to the position
we gave it. This biological fact does away at once with cer-
tain theories found quite frequently in physical and other
textbooks, which speak of the basilar   membrane as con-
sisting of a great number of stretched strings, comparable
to the strings in a piano. These theories assert, after having
introduced, in opposition to the laws of biology, the idea
of a permanent, constant tension of the basilar membrane,
that these different strings—as in a piano—are under different
tension and differently weighted and that they serve accord-
ingly as resonators, responding sympathetically to the va-
rious sounds of the external world.   However pretty this
theory of "the piano in the ear" may appear, authors who
expect their readers to accept it as the truth should first of all
try to convince them of the possibility of living animal tissues
retaining their tension for any length of time instead of ad-

justing themselves to the permanent stretching and thus los-
ing their tension, as all living tissues do. We shall not, of
course, entertain for a moment this idea of the basilar mem-
brane being under constant tension, since our aim is not
unreality, but reality. We need not, therefore, discuss any
further the assumption of the presence of resonators in the
inner ear, which falls with the above rejected, preposterous
assumption of a permanent tension. That the membrane is
capable of resistance, as it probably is, means something
very different from the assertion that it is under constant
tension, which is biologically impossible.

The actual question before us is evidently the question
as to the elasticity of the partition as a whole. Now, we
have seen that the only element of it
which, according to its structure, may be
regarded as elastic, is the basilar mem-
brane. This membrane, however, we have
found to be quite a small part of the bulk

**Is the partition
as a whole
elastic?**

of the partition. If the partition is displaced by an external
force and, this force having ceased, is caused to return to
its original place by the tension which the basilar membrane
has just suffered, such a spontaneous return movement must
be greatly retarded by the bulk of inelastic tissues of the
partition which the membranous part of it has to drag or
shove along with itself. A spontaneous return of the par-
tition to its normal position must be therefore very slow
when compared with the velocity of a displacement caused
by a rather powerful external influence from the stirrup.
Let us, then, keep in mind that with respect to the elastic
properties of the partition there are only two alternatives:
Either the basilar membrane is practically inelastic; then
the partition as a whole is inelastic and cannot sponta-
neously return to its original position after having been
displaced. Or the basilar membrane is elastic; then the par-

tition can spontaneously return after having been displaced, but with a velocity that is only very small compared with the velocity of its displacement.   Of the two alternatives the latter seems to be the more probable.

We saw on a previous page, in our second imaginary case of a partition, that the fluid moves along the unyielding partition, causing friction on the surfaces of the partition. The same friction must be suffered by any part of the real partition as soon as it has reached its upper or lower limit and as long as the stirrup continues to move in the same direction, pushing the fluid on over the initial parts of the partition. If we had to design an apparatus to function thus, would we not see that the surfaces of the partition were sufficiently protected so that the rush of the fluid over them could not injure them? It is interesting to raise this question of protection with respect to the actual partition in the tube. If we look above at figure 7, representing a cross-section of the partition, we notice that the lower surface of the partition is well protected from injury by friction of the fluid by a part of its own structure, the tough basilar membrane. The upper surface, however, with its delicate sensory cells would be exposed to injuries by friction were it not for the membrane of Reissner which we see stretching across the upper division of the tube.   The space between this membrane and the partition does not communicate with the rest of the upper division or with the lower division. It would therefore be really more nearly correct, in speaking of a partition dividing the tube into two divisions which communicate through an opening at the extreme end, to call the total body between the membrane of Reissner and the basilar membrane the partition.   No movements perpendicular to the plane of the

Protection of the surfaces of the partition from the friction of the fluid

drawing can occur in the fluid below the Reissner membrane. The fluid here can only move up and down, pushing or pulling the organ of Corti into its limit of displacement. No friction of the kind above referred to, which might do injury to the delicate tissues of the organ of Corti, can therefore take place, and the problem of protection from friction is thus solved. We shall, however, in order to make our language as simple as possible, restrict the term partition to the organ of Corti, neglecting the membrane of Reissner, since this membrane, aside from the important protection which it offers to the tissues below, does not seem to possess any function whatever.

We saw on a previous page that an imaginary partition which is able to yield to the pressure of the fluid only within certain limits would be exposed to the dan-

The safety valve   ger of breaking whenever an extraordinarily powerful external force tended to cause a movement of the stirrup which would displace more fluid than the yielding partition could make room for, and that this danger might be avoided or at least greatly lessened by an opening of communication between the two divisions at the end of the tube. It is interesting to learn from the researches of the anatomists that such an opening—a safety valve, as we may call it—actually exists at the extremity of the tube of the cochlea.

We may now, after making ourselves familiar with the structural elements of the inner ear and their physical properties, enter into a discussion of the actual function of the organ.

We have thus far taken into consideration only a single movement of the stirrup, in either direction. We must now

Stimulations of the brain resulting from a given rhythmical movement of the stirrup

study the result of a rhythmical movement of the stirrup, back and forth, a number of times during a certain length of time. In order to have a definite case before our mind we will suppose the stirrup to move back and forth in such a way that it will describe a sine curve on a board moving parallel to the plane of the paper. In figure 8 is represented a single period of such a curve in a horizontal position. It is not necessary, however, to imagine this definite curve. What we shall have to say will apply equally to any simple periodic movement, whether of the form of a sinusoid or of a combination of straight lines or of any other

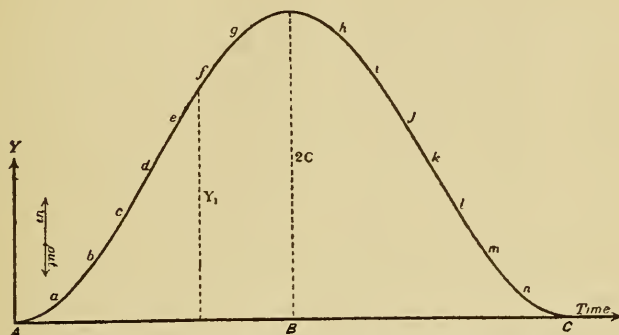Fig. 8.   A curve representing stirrup movement

curve connecting each maximum with the preceding and the following minimum. The question arises then by what means —computation, simple description in words, or otherwise—we can obtain a clear and sufficiently detailed view of the movements of the partition. What we want to know is the form of motion for each point of the partition, and the temporal

relations existing between all the several movements. Only
thus can we obtain a definite view concerning the nervous
stimulations received by the brain as the result of a given
rhythmical movement of the stirrup. In order to find the
movements of the partition in every detail we might try com-
putation since this is the method which yields, although not
always the clearest, yet in general the most accurate results.

Our chief task, then, would be, stated again as definitely
as possible, to find out for each point of the partition which

Computation of the form of motion of the partition

moves at all the exact time which elapses
from a jerk down to a jerk up and from
a jerk up to a jerk down. Figure 9 may
help us to understand the conditions of
computing the time interval in question.
Let us call x the distance of any point of
the partition from the point of x_0 nearest the windows. The
length of the part of the partition which moves in response
to the motion of the stirrup depends, of course, on the ampli-
tude of the movement of the stirrup. This length alone is
represented in the figure. What is farther to the right re-
mains motionless. The dotted lines above and below rep-
resent the upper and lower limit of each moving point of
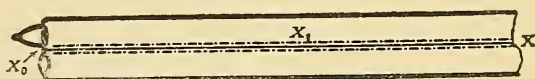
Fig. 9. The partition in the tube and its
limits of movement

the partition. In our curve, figure 8, the minimum, at A, rep-
resents the position of the stirrup most to the left, the max-
imum, at the time B, the position of the stirrup most to the
right. The horizontal line represents, of course, the time. To
the position of the stirrup at A corresponds the position of the
partition (in figure 9) in its upper limit; to the position of

the stirrup at B the position of the partition in its lower limit.
Let us now find out when any arbitrary point x_1 is jerked up
and when it is jerked down, measuring the time from A. It
is obvious that the amount of fluid for which room is made
by the piece of the partition from x_0 to x_1 moving from its
upper to its lower limit is equal to the amount of fluid dis-
placed by the stirrup moving inwards through the distance
measured by y. (For convenience we place the zero point
of the system of coordinates in a minimum point of the curve.)
It would be very easy, therefore, to find the equation of inter-
dependence of x and y, if the following conditions were ful-
filled:

1. If the quantity of fluid displaced were proportional to
the horizontal movement of the stirrup.

Four assumptions provisionally made; not as hypotheses, but for the sake of a gradual compre- hension

2. If the partition were perfectly in-
elastic; that is, not offering any resistance
to a displacement until either of the limits
is reached, and then offering absolute re-
sistance.

3. If the distance between the upper
and lower limits were the same at any
point of the partition.

4. If the width of the partition at any point near the
windows were the same as at any point far away from them.

Let us temporarily regard these conditions as fulfilled. If
they are fulfilled, x is proportional to y. That is, a unit of

Attempt at computation continued

movement of the stirrup always pushes
down (or raises, as the case may be) a
unit of the partition lengthwise. Or, ex-
pressed in a formula:

$$(\text{I}) \qquad y = Cx$$

where C is a constant dependent on the physical properties
of the organ.

The equation of the curve in figure 8 is:

(II) $y = c(1 - \cos 2\pi nt)$;

that is, while t changes from zero to $\frac{1}{n}$, y changes from zero through c, $2c$, and again c, back to zero.   We now substitute Cx for y:

$c(1 - \cos 2\pi nt) = Cx$, consequently:

(III) $\cos 2\pi nt = 1 - \frac{C}{c}x$

This formula permits us to calculate t, that is, the exact time when any point of the partition is jerked down.   But it holds good only for the time from A to B, that is, while the stirrup moves in one direction.   As soon as the stirrup reverses its movement a new formula has to be applied, since the movement of the partition is of a kind which is mathematically called a discontinuous function.   The moment when the stirrup reverses its movement and the farthest point of the partition has been jerked down, the function jumps, so to speak, from this point to the beginning of the partition and the first point, nearest the windows, is jerked up.   The formula to be used from B to C is to be derived by substituting $(2c - y)$ for y in (I), since x would now be proportional to $(2c - y)$. We then have the following new equations:

(IV) $2c - y = Cx$.

(II) $y = c(1 - \cos 2\pi nt)$, consequently:

(V) $\cos 2\pi nt = \frac{C}{c}x - 1$.

This formula is valid from B to C, that is for values of t varying from $\frac{1}{2n}$ to $\frac{1}{n}$, while the partition is being jerked upwards.   We notice that the only difference between the right side of (III) and the right side of (V) is the sign. For the same x we obtain the same absolute value of $\cos 2\pi nt$, but in the one case it is positive, in the other negative.   Now, it is easy to see what this means for the time interval between a downward and an upward jerk of any point of the partition.

Remembering that (III) is valid for jerking down, (V) for jerking up, we notice that the arc of $\cos 2\pi nt$ runs through the first and second quadrant while the partition is being jerked down, through the third and fourth quadrant while the partition is being jerked up. Therefore, since we found that the time of jerking down of a definite point x_1 and the time of jerking up of the same point are subject to the condition that $\cos 2\pi nt$ yields the same absolute value, but differing in sign, the time of jerking up must be found in a quadrant opposite to the quadrant wherein the time of jerking down occurred, never in an adjoining quadrant; that is, if the former time is to be found in the arc $2\pi nt$, the latter must be found in the arc $2\pi n(t + \frac{1}{2n})$, since the addition of $\frac{1}{2n}$ to t means the addition of two quadrants. The difference of time, therefore, is always $\frac{1}{2n}$. In other words, the time interval from a jerk down to a jerk up and from a jerk up to a jerk down of any definite point is with this particular curve always the same, being exactly one half of the whole period. We have thus found by computation the exact movement of the partition in case the movement of the stirrup is of the form of a sinusoid.

We have seen then that, provided a certain set of conditions (our four provisional assumptions) is fulfilled, and provided the movement of the stirrup is of the form of a simple sine (or cosine, as this means the same) curve, computation of the movement of the partition is possible. But computation is neither particularly clear—at least those who are not professional mathematicians will think so—nor is it universally applicable, but only in a few cases of stirrup movement, the above, the case of straight lines connecting the maxima and minima, and a very small number of others.

Summary of the foregoing discussion

To prove that computation is not universally applicable let the movement of the stirrup be represented by the function

$$y = c(2 - \cos 2\pi mt - \cos 2\pi nt)$$

Computation abandoned

and let m be equal to 4 and n equal to 5 (the simple case of a major third, musically speaking). Even in a case like this, by no means far fetched, rather the contrary, computation is impossible since it would involve, as the mathematical reader may easily convince himself, the solution of an equation of the fifth degree in order to find the mutually corresponding values of y and t for the maxima and minima of the curve. Without these values for the maxima and minima, which are the points of discontinuity of the function representing the movement of the partition, we could not proceed at all. It is out of the question, therefore, to expect that computation pure and simple, even under the four assumptions provisionally made, will ever give us a satisfactory comprehension of the function of the inner ear. We must look for other means in order to obtain our end, an insight into the details of movement of the partition.

Let us, then, try to represent the movement of the partition in the above case as well as in others graphically. I

Graphic methods of determining the exact movement of the partition

shall offer to the reader two methods of graphic representation. The first of these is more accurate in some respects than the second, but a little more difficult of application.

The vertical axis of our system of coordinates in figure 10 may represent the succession of points of the partition, beginning from next to the windows. The

First graphic method

horizontal axis may represent the time. I must warn the reader against thinking that the figures resulting on the paper are pictures of something that exists in the ear or elsewhere. The

figures are not pictures of existing things but merely symbols of a function, that is, of the time when any point of the partition is jerked up or down. The construction of the figure is based on the following considerations. Let us mark on the paper the points indicating the time when any given point of the partition is jerked. When we shall have marked a sufficient number of such points, we shall draw a curve through them. But how do we find the points? The movement of the stirrup is represented in figure 8. When the stirrup has its extreme position to the left (according to Fig.
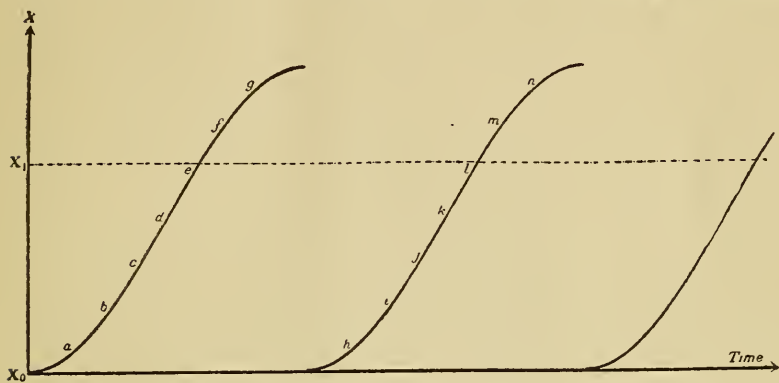
Fig. 10. Graph of the times when each point of the partition is jerked down (curves of odd numbers) and up (curves of even numbers). Compare figure 8

9) and just begins to move inwards, we mark the time as zero and the point of the partition which is jerked down also as zero, since the point which is jerked is the point nearest the windows. In figure 10 we find this point near *a*. As the time advances (Fig. 8) the stirrup moves farther and farther inwards, with gradually increasing and later again decreasing velocity. A further point, say *b*, in figure 10 must be located somewhat to the right of *a* and above *a*, since a more distant point of the partition is represented by a higher position of the mark in our system of coordinates, and the

fact that it is jerked later is represented by a position farther
to the right. Now, since the velocity of the stirrup increases
as shown by figure 8, the following marks have to be placed
higher than proportionate to their advance to the right. That
is, points marked off by equal steps on the partition are now
jerked successively in briefer time intervals than before.
Later approaching the time B in figure 8, the stirrup moves
again more slowly, and the marks in figure 10 advance there-
fore more rapidly towards the right, as seen in f and g. If
we now draw a complete curve through the marks a, b, c, d,
e, f, g, we convince ourselves readily that the new curve is
the same curve as the one in figure 8 from A to B. Of course,
if we have not chosen the same vertical and horizontal scales
in both figures, the new curve must appear more or less steep
than the old one. But the selection of a scale for a graphic
representation is entirely a matter of convenience. Choosing
identical scales, we simply have to transplant the first half
of the curve in figure 8 from A to B into the new figure.

But now the stirrup begins to move in the opposite di-
rection, causing the partition to be jerked upwards gradually.
The point of the partition nearest the windows is jerked up
first, the others later in regular order. Now, it can be easily
seen where we have to place the further marks in our new fig-
ure, namely h, i, j, k, l, m, n. We find them, or rather immed-
iately the complete curve of which they are points, by simply
turning the second half (B to C) of the curve in figure 8 up-
side down, without, however, making any change between
right and left. In this way we go on, simply transplanting the
parts of the stirrup curve, leaving the rising ones in the same
position, but turning the falling parts upside down.

If we now desire to find out for any point of the parti-
tion, for example, for x_1, the exact time when it is jerked
down and when it is jerked up, all we have to do is to pass
on from this point (Fig. 10) to the right (along the dotted

line), since this direction, according to definition, represents
the time. Our first crossing of a curve (in *e*) means a jerk
down; the next crossing (in *l*) a jerk up; and so forth. That
is, the odd crossings mean each a jerk down, the even crossings
each a jerk up. The time intervals can then be measured
with a rule. We find in this special case that the intervals are
all equal. We have thus graphically represented the exact
movement of the partition in a case where the movement of
the stirrup is of the form of a sinusoid. The same graphic
representation is applicable to any given curve, however com-
plicated it may appear. This method has universal validity.
We shall soon convince ourselves of its importance for the
analysis of a complicated curve.

We can easily learn from the graphic representation be-
fore us that under the assumptions provisionally made the

**What movement
of the stirrup
produces the
sensation of a
single tone (free
from overtones)?**
stimulation of each nerve ending can
hardly be influenced by the form of the
stirrup curve, that is, whether this curve
is a sinusoid, or made up of straight lines
connecting the maxima and minima, or of
any other shape, provided the maxima and
minima remain unaltered. Let us sup-
pose that each "down" means a shock to
the nerve end and that the "ups" are indifferent as to ner-
vous excitation. We see immediately (Fig. 10) that the time
interval between two shocks at any point of the partition
must be exactly the same, since each down curve would be
exactly like any other down curve, whatever the shape of
the up curve. (This result would be the same if the "ups"
meant excitation of the nerve end and the "downs" were in-
different.) That is, the particular shape of the curve rep-
resenting the movement of the stirrup, has no significance
for the question whether a single tone will be heard or not.
If all the down curves are identical, a single tone only is

audible. I remind the reader, however, that we are deriving this conclusion on the basis of our provisional assumptions, and further, that we are speaking here of movements of the stirrup, not of rhythmical pressure changes of the air in the external ear or of movements of a tuning fork or any other vibrating body. In discussing later the effect of the latter conditions upon the stirrup, we shall see that their form is not necessarily identical with the form of the stirrup movement.

As yet, we have studied only very simple movements of the stirrup. Before we take up the problem of how the inner ear analyzes more complicated move-

The physiological condition of tone intensity
ments of the stirrup, we ought to remember that we have not yet discussed the physiological condition of tone intensity. We have spoken only of the frequency with which shocks are received by the nerve ends. But the frequency of the shocks determines only the attributes of pitch and quality, not the attribute of intensity of a tone sensation. Let us look to another sense organ, the olfactory organ, for a suggestion. On what physiological condition does the intensity of an odor depend? Although we have no definite knowledge here any more than in the sense of hearing, we have reason to believe that the intensity of an odor depends, or may depend, on two conditions: 1. The number of nerve ends stimulated; and 2. the concentration of the substance which stimulates each of these nerve ends. Accepting this suggestion we have to see what conditions might determine tone intensity. Only these two can come up for consideration, so far as I can see: 1. The number of nerve ends which receive shocks in a definite frequency; and 2. the suddenness, the impetuosity with which each nerve end is shaken when the point of the partition in which it is located is jerked down. Now, the second of these two conditions

is theoretically almost beyond our reach. We cannot, in the present state of our knowledge, obtain a very clear idea of differences in the suddenness with which the nerve ends might be shaken in different cases. It will be best, therefore, to omit this factor in the discussion of intensity altogether, or at least for the present, rather than burden our theory with arbitrary hypotheses the usefulness of which is no more probable than their uselessness. At present we shall limit our discussion to the first condition, the number of those nerve ends which are stimulated with equal frequency.

It is clear that the number of nerve ends stimulated depends in some way on the length of that part of the partition which is jerked up and down in a certain frequency. But here we are confronted by this difficulty. We do not know whether the nerve fibres are equally distributed along the partition. It might be the case that on a certain length of the partition near the windows a greater number of nerve ends were found than on an equal length farther away from the windows; or the reverse. In our present state of knowledge this difficulty cannot be overcome. In order to go on with our theory, we have to make an assumption. We shall make, of course, the simplest, the least arbitrary assumption. We assume, provisionally, that equal parts of the partition lengthwise contain equal numbers of nerve ends. If it should be found that the theory agrees with the facts of auditory observation more closely under another assumption, we would have to substitute this for the one now made. Of course a definite answer given to the problem by the anatomists would be more satisfactory.

A difficulty in the theoretical determination of tone intensity. Fifth provisional assumption

We can measure the length of that part of the partition which is jerked up and down, only by the aid of our knowledge (if we have any) of the movement of the stirrup. Now, the reader will recall among our provisional assumptions the one that the width of the partition at any point near the windows is the same as at any point far away from them. But the anatomists tell us that this assumption is incorrect; that the partition is about twelve (or more) times as wide at the end as near the windows. Nevertheless we shall provisionally make the assumption of proportionality between any length of the partition being jerked up and down and the extent of the movement of the stirrup which causes the movement of this piece of the partition, in order to understand first a simpler, though imaginary, case and to proceed gradually to a comprehension of the actual, rather complicated function of the partition. Let us be aware, however, that, having thus simplified the actual conditions, we cannot expect to find a perfect, but only an approximate harmony between the results of a theoretical analysis and the direct observations of an actual sound analysis by the ear. We may find, indeed, with respect to tone intensity, rather serious disagreements between the facts and the theory. But these disagreements will disappear as soon as the theory takes account of what, for simplicity's sake, we provisionally neglect.

Another difficulty in the theoretical determination of tone intensity

Making the two provisional assumptions just mentioned, we can theoretically measure the intensity of a tone sensation by the total length of that part of the partition the nerve ends of which are excited with one definite frequency. In our graphic representation (Fig. 10) the intensity can then be measured by the vertical distance between the horizontal coordinate and the top of the curves which represent the down and up jerks.

Tone intensity in our graphic representation

We discussed above the result of a simple back and forth movement of the stirrup. Let us now do the same with a more complicated movement. Figure 11 represents the new stirrup movement which we are going to study. This curve is approximately the one represented by the equation

$$y = (1 - \cos 2\pi 2t) + (1 - \cos 2\pi 3t);$$

which justifies us in saying that it represents physically the sum of two tones of the vibration ratio $2:3$. Let us apply

Analysis of the combination 2 and 3
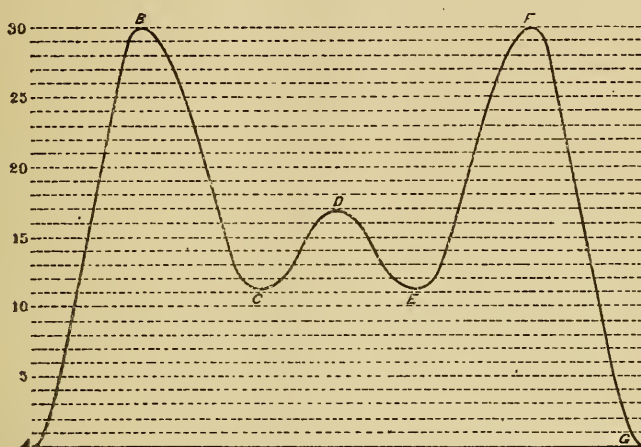
Fig. 11. The combination 2 and 3. First characteristic phase

the same graphic method to this case. We have first to transplant the part of the curve from the first minimum to the following maximum, A to B, into figure 12. Now, when the stirrup reverses its motion, the parts of the partition near the windows begin to be jerked up. Therefore, the curve from the maximum B to the next minimum C has to be turned upside down and then transplanted. The following part of the curve, from C to D, must be transplanted in its original upright position, but placed on the

horizontal coordinate of the new figure, whatever its elevation in
the original curve may be, since every reversal of the movement of
the stirrup causes at once a movement of the parts of the parti-
tion next to the windows and only later a movement of the follow-
ing parts.   So we continue transplanting each section of the
curve, alternately upright and upside down.   This figure (Fig. 12)
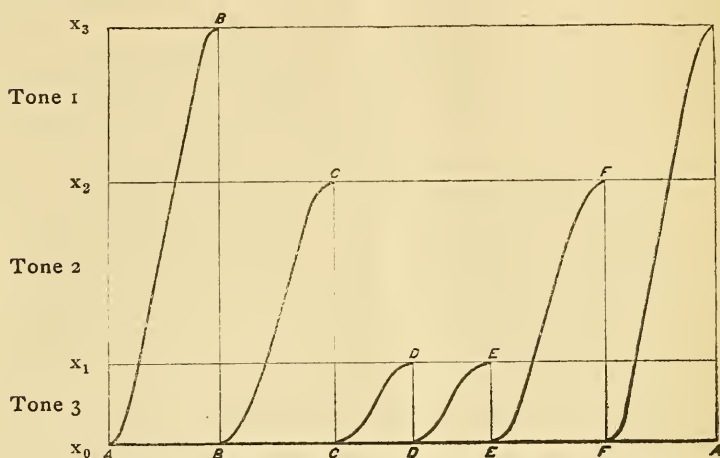
Figure 12. The combination 2 and 3. First characteristic
phase. (A is identical with G.) Compare figure 11

is to be interpreted in the same way as figure 10.   The distances
from x_0 to x_1, x_1 to x_2, and x_2 to x_3 represent three pieces
of the partition, x_0 being next to the windows.   During the
unit of time, which is here the period from A to G, all the
nerve ends located between x_0 and x_1 receive, as is easily
seen, three shocks, counting the number of shocks received
by the number of downs (or ups, since this distinction between
the physiologically effective and ineffective direction of jerk-
ing is arbitrary, for want of better knowledge as to the man-
ner of excitation of the nerve ends).   All the nerve ends be-
tween x_1 and x_2 receive, as the figure shows, counting from
left to right, two shocks in the unit of time.   And all the

nerve ends between x_2 and x_3 receive one shock. The nerve ends located farther towards the apex of the cochlea do not receive any stimulation and do not, therefore, concern us. How many tones should we expect then to hear in this case? The answer is as easy as simple: Three different tones, since shocks of three different frequencies are received by the several nerve ends. And the musical relationship, the pitch, as we say, of these tones is determined by the relative frequencies found, which are 3 and 2 and 1. The relative intensity of these tones is to be measured, in accordance with our remarks in the preceding paragraph, by the relative lengths $x_0\,x_1$, $x_1\,x_2$, and $x_2\,x_3$.

A movement of the stirrup, not probably exactly like, but similar to the one just discussed could be produced by sounding simultaneously with approximately equal intensities two tuning forks representing the ratio of vibration rates 3:2. It is well known that we hear in such a case three different tones, 3 and 2, which we may call "objective" or primary tones, and 1, which we may call a "subjective" or difference tone. Some further facts concerning such subjective or difference tones will be mentioned subsequently for those readers who are not familiar with the conditions under which they make their appearance. The appropriateness of calling the subjective tones in question "difference tones" will then become apparent. The fact that our theory of the function of the inner ear and actual observation in this case agree so nicely, is highly satisfactory to us and ought to encourage us to proceed further in applying the theory to other special cases of movements of the stirrup. Let us keep in mind that our theory thus far has explained in a special case two most fundamental observations: 1. That our organ of hearing is capable of analyzing a compound

Two important facts: Sound analysis and production of subjective difference tones

acoustic process; and 2. that it has the power of producing
on its own account subjective tones which no study of mere
external conditions could ever have revealed to us as a natural
consequence of the physical processes we call tones.

We saw in the preceding paragraph that all the nerve
ends between x_0 and x_1 received three shocks in the unit of
time. A measurement of the distances in
A problem for   the figure, however, shows that the time
future solution   intervals between these three shocks, al-
though approximately the same, are not
exactly alike (and, moreover, there are differences in this re-
spect between the several nerve ends all of which receive three
stimulations). Now, it is probable that the particular nervous
excitation set up in each ganglion cell by these three stimula-
tions of its terminal fibre and thence carried farther to the
brain, may be just the same in either case, whether the shocks
are received in an exactly regular rhythm or in a slightly irreg-
ular succession. It will be one of the problems of the future
to decide what is the limit of irregularity which must not be
overstepped if the sensation produced is to be the same as that
of a regular series of shocks of the same frequency. At pres-
ent we have hardly any certain data upon which to found a
decision. We must leave this problem open for the present.
It would be well, however, to remember that the above graphic
representation of the movement of the partition—for simplic-
ity's sake—is based on a number of assumptions, and that
the actual movement of the partition is doubtless somewhat
different from the one which is here under discussion, and
which contains probably only the essential features of the
actual movement, not all its minor details. It is entirely pos-
sible, under these circumstances, that the irregularity in ques-
tion is in reality much less considerable than it appears to us
now, and what seems to be an important problem, may turn
out to be no problem at all. The reason we have for believ-

ing that the actual irregularity might be less than the one found here, is that in the graphic representation we have assumed a movement made up of absolutely sudden, unprepared jerks, with intervals of perfect rest between them. The real movement is probably a more gradual change from rest to motion and back to rest; and the result of this might very well be an equalization of the time intervals preceding the shocks received by the nerve ends. This, however, is not offered as a solution of the problem, but merely as a suggestion for the future investigator of this subject.

Let us try another method of graphically representing the movement of the partition under the provisional assumptions made. This method has a certain disadvantage as compared with the method used above, in being less accurate with regard to the time intervals, but, on the other hand, the advantage of a greater simplicity for the constructor as well as for the reader. The extension of the partition from the windows towards the apex of the cochlea is here represented, not—as before—by the vertical, but by the horizontal extension of the figure, from left to right. Figure 13 shows the method as applied to the same curve (Fig. 11) which we have just discussed. The first thing we have to do is to draw in the given curve (Fig. 11) at equal distances so many lines parallel to the horizontal coordinate that each of the maxima and minima can be regarded as lying on one of these parallels. If this is not easily done, then any arbitrary number of parallels may be drawn. But the drawing as well as the interpretation of the new figure requires a little more attention in this case, because we have to consider fractions. In this figure there are thirty equidistant lines drawn parallel to the horizontal coordinate. A greater accuracy than this would be entirely out of place, since our representation in any case

Second method of graphic representation of the movement of the partition

is merely an approximate representation of the actual move-
ment of the partition. These horizontal parallels are auxiliary
lines, serving the purpose of a measuring scale. The second
thing we have to do is to draw a second, independent, system
of auxiliary lines enclosing a corresponding number of spaces.
These lines are the thirty-one vertical parallels in figure 13.
The horizontal lines here indicate for the times A, B, C, and
so forth, the positions of the different points of the parti-
tion at the upper or lower limit of movement. The vertical
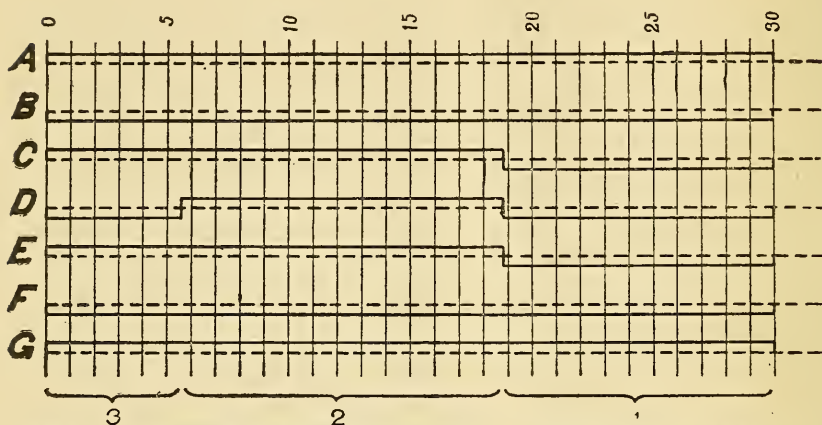
Fig. 13. Successive positions of the partition. The combination 2 and 3.
First characteristic phase. Compare figure 11

auxiliaries serve the purpose of cutting off the partition a
number of equal sections corresponding to the number of parts
into which we divided the total amplitude of the given curve
representing the movement of the stirrup. To the right of
these sections which move are to be imagined the parts of the
partition nearer the apex which do not move at all in this spe-
cial case and which do not, for this reason, concern us here.
At the time A, all the moving parts of the partition are at
their upper limits, since the stirrup has at this time its extreme
outward position. From A to B, the stirrup moves through

thirty units inwards, pushing down successively all the thirty
sections of the initial part of the partition. We find, therefore,
in figure 13 at B all the thirty sections at their lower limits.
From B to C, the stirrup makes an outward movement through
nineteen spaces. The result is an upward movement of an
equal number of sections of the partition. We find, therefore,
at C the first nineteen sections of the partition at their upper
limits. All the following parts of the partition remain ex-
actly in the positions at which they were at the time B, since—
according to the assumptions under which we are working—
no force whatsover has acted upon them. That is, the sec-
tions twenty to thirty are still at the lower limits, and the
further parts of the partition in their normal positions. From
C to D the stirrup moves inward through six spaces, as seen
in figure 11. It causes therefore the first six sections of the
partition to be jerked down. In this position we find them
in figure 13 at D. All the rest of the partition remains exactly
as it was at C. That is, the next thirteen sections are still
at the upper limits and the following eleven still at the lower
limits where we found them at B. From D to E, the stirrup
makes an outward movement through six spaces, causing an
equal number of the initial sections of the partition to be jerked
up. We therefore find in the figure at E the first nineteen sec-
tions of the partition at the upper limits, the following eleven
at the lower limits. From E to F, the stirrup moves inward
again through nineteen spaces, causing nineteen sections of
the partition to be jerked down. We find, therefore, in the
figure at F all the thirty moving sections of the partition at
the lower limits. From F to G, the stirrup moves outward
through thirty spaces, as seen in figure 11. This causes thirty
sections of the partition to be jerked up. So we find in figure
13 at G the whole initial piece of the partition which moves
and therefore alone concerns us, at the upper limit. The stir-
rup has now reached the very position from which it started

at A; and the partition has the same position which it had
then. We have thus graphically represented the characteristic
positions through which the partition passes during a com-
plete period of the movement in question.

The graphic representation, of course, is only a means
to an end. We have to read off from this representation
how many shocks are received during the
period by the nerve ends on each section
of the partition. This is easily done. Let
us again, for want of definite knowledge,
make the assumption that a jerk down of
the partition means a stimulation of the
nerve ends, and that a jerk up is irrelevant. We then sim-
ply have to go down in the figure from the top to the bottom
and count the number of times each section is jerked down.
The first section is down at B, up again at C, down for a sec-
ond time at D, up again at E, down for a third time at F,
and up again at G. The nerve ends on this section, there-
fore, receive three shocks during the period. We find the
same number of stimulations on the following five sections.
Let us now inspect the seventh section. It is down at B, up at
C and still up at D and E. It is down for a second time
at F and up again at G. That is, the nerve ends on this sec-
tion receive two shocks during the period. The same is true
for the following twelve sections. Let us now look at the
twentieth section of the partition. It is down at B, still down
at C, D, E, and F; up again at G. That is, the nerve ends here
receive only one shock during the period. The same holds
for the following ten sections. We see, then, that three
tones must be simultaneously heard, which we may call,
according to the relative frequency of stimulation, the tones 3,
2, and 1. The relative intensities of these tones may be re-
garded—under the provisional assumption of a uniform dis-
tribution of nerve ends lengthwise over the partition—as six,

How to read off
the tones heard
and their
intensities

thirteen, and eleven, according to the number of sections which receive the greater or smaller number of shocks.

Let us now apply the second graphic method to another given movement of the stirrup, which will make clear to us

Difference of phase. Characteristic curves of a tone combination

another interesting property of the ear with respect to the manner in which this organ analyzes an objective sound. The curve of the stirrup (Fig. 14) is made up of two component curves, very similar to the curves composing the last curve discussed. That is, each of the two components is approximately a sinusoid, one of a period equal to two thirds of the other's period, both of approximately the same amplitude. The resultant curve is constructed here as before by measuring and adding together the ordinate values of the components in the drawing. The difference between the present case and the last case discussed is a difference of phase. If the reader should not know what this means, it can be easily understood by the aid of figure 14. We find there two sinusoids, one with two and one with three maxima within the same period, which accordingly may be called curve two and curve three. Now imagine curve two moved slightly to the right until the minima at the extreme right and also the minima at the extreme left coincide. We then have exactly the case discussed above; that is, the addition of the two curves would result in a compound curve as represented by figure 11. The curves of figure 11 and of figure 14 may be called the characteristic curves of the ratio 2:3, because they are the two extreme forms between which the compound curve changes as the result of a change of phase, that is, of a lateral movement of curve two, while curve three remains stationary. Let us convince ourselves here that there are no more than two characteristic compound curves. If we move curve two again slightly to the right, the same distance as before, that is, one twelfth of the

period, we obtain a compound curve as shown in figure 16, which is exactly like figure 14 when read from the right to the left. And if we change the phase again in the same manner, that is, move curve two again one-twelfth of the period to the right, we obtain a compound curve as shown in figure 18, which is exactly like figure 11 only turned upside down. We shall demonstrate in the succeeding paragraphs that it is entirely irrelevant with respect
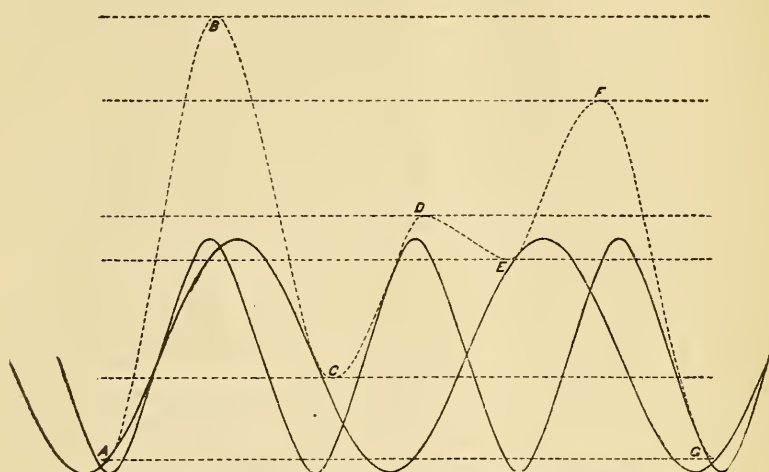
Fig. 14.   The combination 2 and 3.   Second characteristic phase

to our theory whether we read a curve from the left or from the right, in its first position or turned upside down. We shall demonstrate thus that there are indeed only two compound curves, no more, which are characteristic of a combination of two sinusoids. This is an important fact because it makes much simpler and easier our task of comprehending the function of the inner ear.

Let us apply, then, the second graphic method to this second characteristic curve of the combination 2 and 3. We

Theory applied to second characteristic curve of combination 2 and 3

locate, in figure 14, the horizontal coordinate so that the absolute minima of the compound curve are to be found thereon. We then draw a number of equidistant lines, say thirty, parallel to the horizontal coordinate. To avoid making the figure obscure I have indicated of these parallels only those which pass approximately through the maxima and minima of the curve. We further draw a system of thirty-one equidistant vertical parallels enclosing a series of thirty equal spaces which represent succeeding pieces of the partition. In this system of auxiliaries we represent the positions of the partition at the time A, B, C, and so forth. At A in figure 15 we find all the moving sections of the partition at their upper limits, since the stirrup has at this time, as figure 14 shows, the most outward position, the external air pressure and accordingly the density of the air in the middle ear being lowest. At B we find all the thirty initial sections of the partition down,
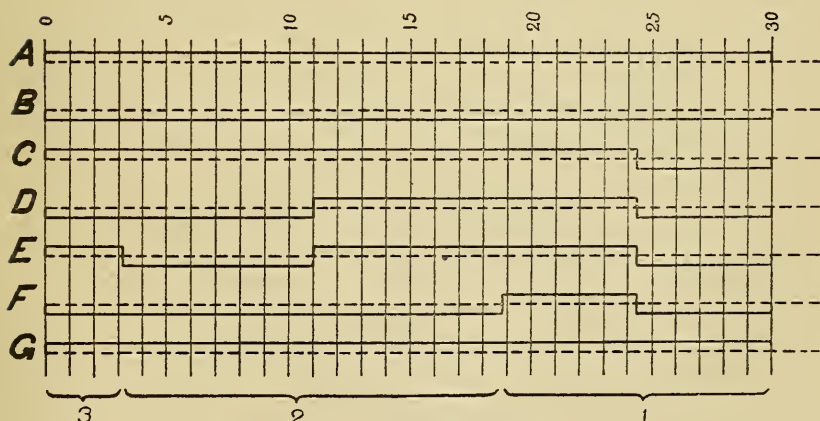

Fig. 15. Compare figure 14

since from A to B the stirrup has moved through thirty units of space inwards. At C we find the twenty-four initial sections raised again since the stirrup has moved outward through twenty-four spaces. At D the eleven initial sections of the partition are at their lower limits since from C to D the stirrup has moved through eleven spaces in an inward direction. From D to E the stirrup moves outwards through three spaces. Accordingly we find at E the first three sections of the partition raised to their upper limits. From E to F the stirrup moves inwards through eleven spaces. Accordingly eleven sections of the partition must be pushed down to their lower limits. We find the first three down at F. The following sections up to the twelfth were already down at E. In order to represent eleven sections of the partition as just pushed down we have to place at F the twelfth and the following, including the nineteenth, sections of the partition at their lower limits. Then the first three and the latter eight make up the total number of eleven sections pushed down. From F to G the stirrup moves outwards through twenty-four spaces. Accordingly all sections of the partition are raised to their upper limits except those from the nineteenth to the twenty-fifth which were already at their upper limits at F and therefore simply stay there. So we find the partition at the time G in exactly the same position in which it was at A; and we must find it again in the same position since now another period of stirrup movement begins, exactly like the period just discussed. We now have to read off the tones heard and their intensities in the same manner as we did this before. The result is that we must expect to hear the three tones 3, 2, and 1 in the relative intensities three, sixteen, and eleven.

Comparing our analysis of the curve in figure 14 with the
former result obtained from figure 11, we observe that in

Practical -
irrelevance
of phase

spite of the remarkable difference of ap-
pearance of these curves to the eye, the
tones which we expect to hear are the
same. This is, of course, of the greatest
importance in musical practice. Imagine
the unsurmountable difficulties if the director of an orchestra
were responsible for the phase in which the several tones pro-
duced by the members of the orchestra acted upon the audi-
tory organs of each hearer in the concert hall. But, as it is,
each hearer perceives the same tones whatever the phases of
the objective processes in the air. Now those who believe
in the existence of a system of strings like "a piano in the
ear," have laid much stress on this fact of the practical irrel-
evance of phase, and some have even gone so far as to say that
it compels us to assume sympathetic resonance to be the me-
chanical power of the auditory organ. I need not persuade
the reader, however, that such a compulsion does not exist.
Some have gone still farther and asserted that phase differ-
ence has never and under no circumstances any influence
whatsoever upon the auditory perception. Their theory of the
mechanics of the inner ear may lead to such a consequence, to
an absolute irrelevance of phase. Experiment, however, has
not yet proved that phase difference of the sinusoidal compon-
ents of stirrup movement has never any influence of any kind
upon the perception. Our theory has shown us the practical ir-
relevance of phase differences and, at the same time, left a pos-
sibility for slight influences of this kind upon the perception,
resulting in a change of the relative intensities of the sev-
eral tones heard. The intensities of the three tones
for one phase we found to be six, thirteen, and eleven; for the
other phase three, sixteen, and eleven. That is to say, we
would hear in the second case the same tones, but their relative

intensities would not be exactly the same as those in the first case. That is, difference of phase may be irrelevant, but it need not be so. Let us recall, however, that our representation is only a rather remote approximation to the actual movements of the partition, so that actually the influence of phase upon the perception may be other than it here appears to be. What is important is our insight into the possibility of a slight influence of this kind.
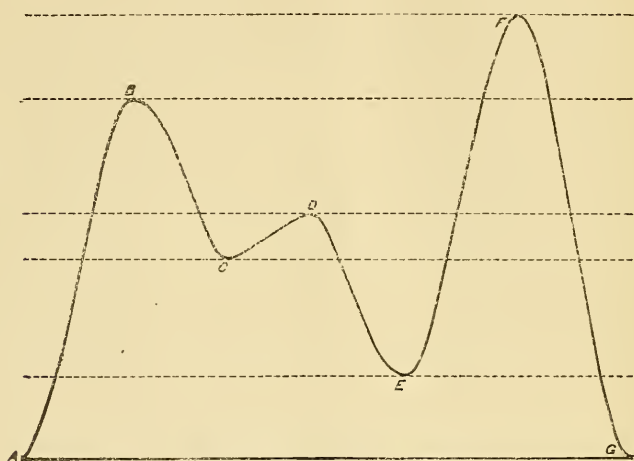
Fig. 16.   Compare figure 14

I promised to demonstrate that the application of our theory yields the same result if we read the curve of stirrup movement from the right to the left, or turn it upside down. The former case is illustrated by figure 16, which is exactly like figure 14 when read from the right to the left. Figure 17 shows the successive positions of the partition. At B the twenty-four initial sections are down. At C the first eleven of them

Theoretic irrelevance of the sign of the coordinates

are up again. At D three are down again. From D to E the
stirrup moves through eleven units of space outwards.
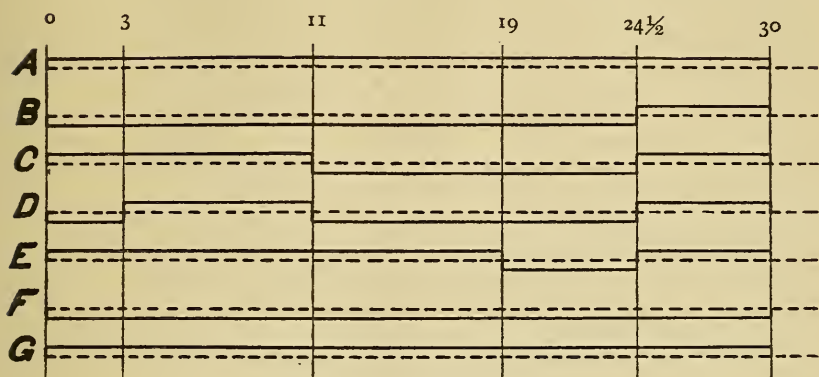
Fig. 17.   Compare figure 16

Therefore at E the first nineteen sections are up, eight of them
being up already at D. From E to F the stirrup moves in-
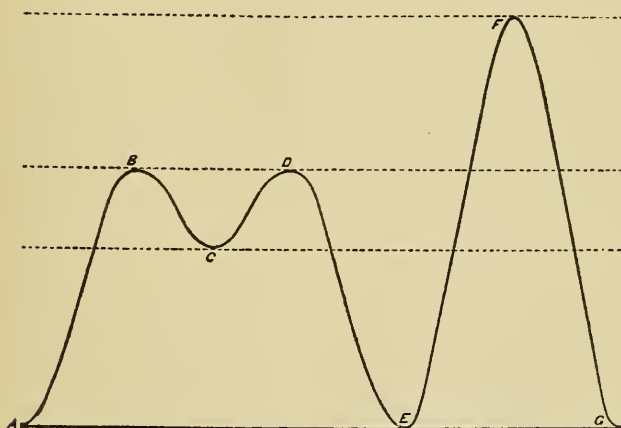
Fig. 18.   Compare figure 11

wards through a little more than twenty-four units of space.
Therefore at F thirty sections are down, five of them being

down already at E. At G (equal to A) all the thirty sections
are up again. The tones to be heard, which the reader after
all the previous practice in this task can easily read off, are
3, 2, and 1 with the relative intensities three, sixteen, and eleven.
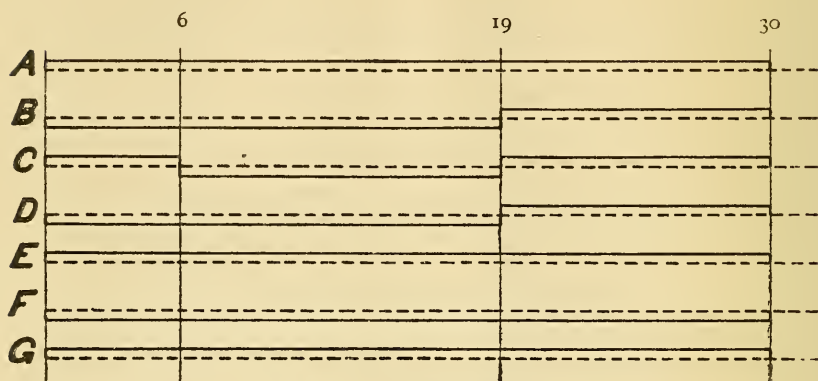
Fig. 19.   Compare figure 18

As expected, this result agrees perfectly with our analysis of
the curve in figure 14.

Let us now demonstrate that turning the curve upside down
has no influence on the theoretic result. Figure 18 is exactly
like figure 11, only turned upside down. In figure 19 we see
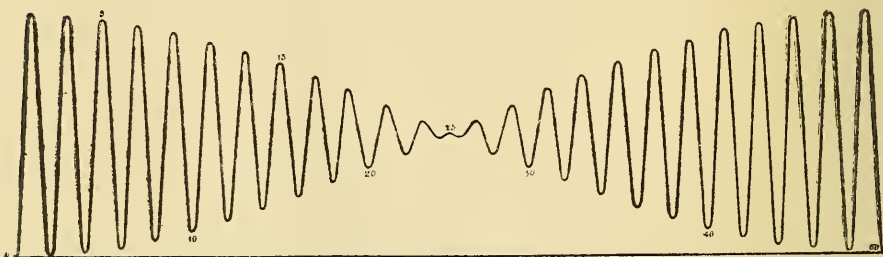
Fig. 20.   The combination 24 and 25
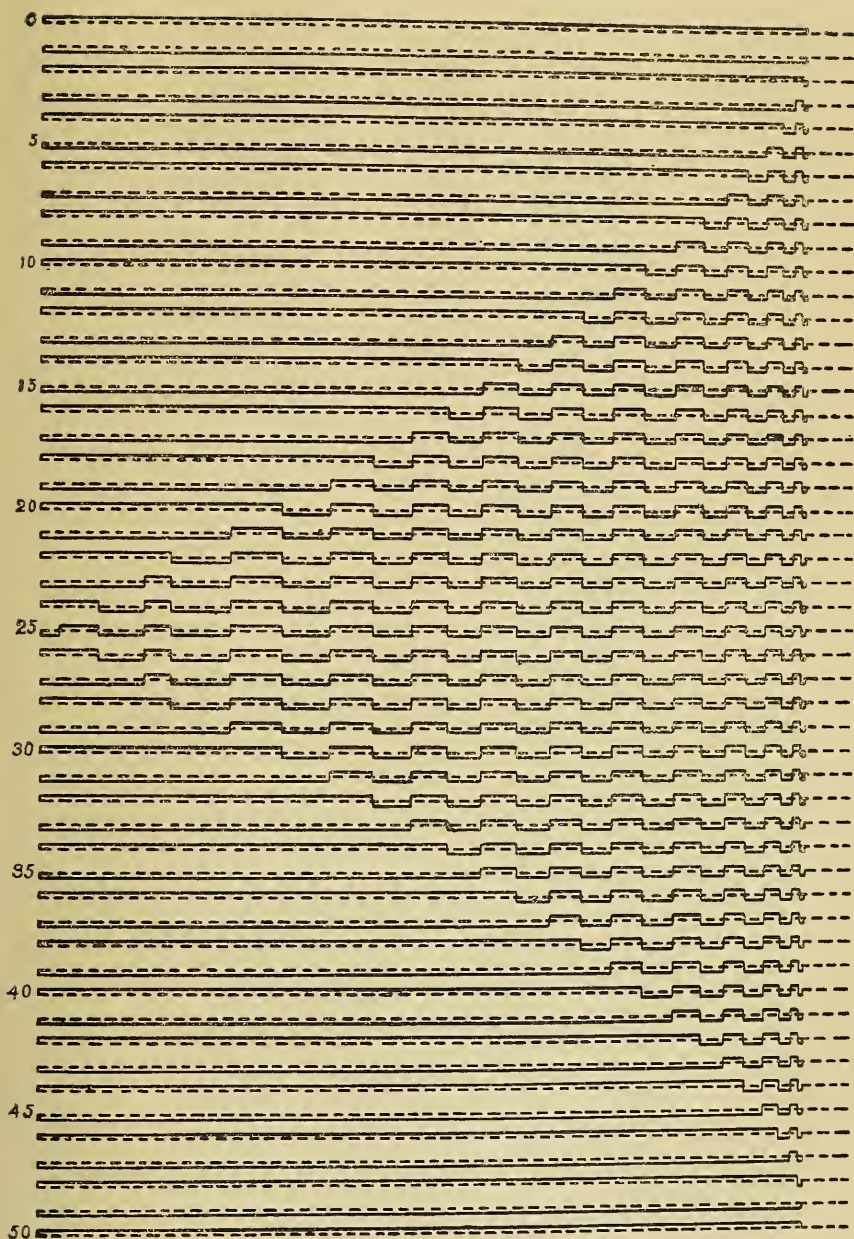
Fig. 21. The combination 24 and 25. Compare figure 20

the successive positions of the partition corresponding to this curve. The interpretation of the figure is so simple that the reader will easily read off, without any aid, what tones are to be heard; namely the tones 3, 2, and 1 with the relative intensities six, thirteen, and eleven. This is exactly the same result as that of our analysis of the curve in figure 11.

The interval studied above is in musical terminology that of a fifth. Let us now study an interval which is even smaller than a semitone. The compound curve in figure 20 is made up of twenty-four vibrations originating from one source and twenty-five from another. Figure 21 shows the successive positions of the partition corresponding thereto. The initial section of the partition moves up and down twenty-five times during the period. We may, therefore, conclude that the nerve ends located here will transmit to the brain a process resulting in the sensation of the tone 25. In order to discuss this matter with more accuracy, I have not relied only upon the draftsman's skill in constructing the compound curve, but computed the ordinate values of some of the maxima and minima. Such a computation is exceedingly tiresome work, since for each pair of values in the table it is necessary to compute twenty or more values in order to select from them what appears as the maximum or minimum. But the accuracy of this method can be carried to any decimal desired. We learn from the table of these values that the relative intensity (when determined in the same way as above) of the tone 25 would be nine (that is, 200—191).

The tone combination 24 and 25

INTERVAL 24:25, EQUAL AMPLITUDES

	Abscissa	Ordinate	Abscissa Difference	Point	Ordinate Difference
Min.	0	0	73	0	400
Max.	73	400	73	1	400
Min.	147	2	74	2	388
—	—	—	—	—	—
Max.	1540	246	—	21	—
Min.	1620	167	80	22	79
Max.	1685	221	65	23	54
Min.	1755	191	70	24	30
Max.	1800	200	45	25	9
Min.	1845	191	45	26	9
Max.	1915	221	70	27	30
Min.	1980	167	65	28	54
Max.	2060	246	80	29	79
—	—	—	—	—	—
Min.	3453	2	—	48	——
Max.	3527	400	74	49	388
Min.	3600	0	73	50	400

If we regard—quite arbitrarily—the time from one stimulation to the next as measurable by the abscissa differences

Do we hear the tone 25?

of the succeeding maxima, we observe that this difference is about one hundred and forty-seven at the beginning of the period, that it decreases very slowly and is about one hundred and forty-five at the maximum twenty-three, about one hundred and fifteen at the maximum twenty-five, the same at the maximum twenty-seven, and that it increases gradually till the end of the period. One twenty-fifth of the whole period is one hundred and forty-four. This is the average abscissa difference, on which the pitch of the tone heard depends, since the abscissa difference is inversely proportional to the frequency of stimulation. But the actual abscissa differences, as we

have just seen, deviate from the average, particularly in the middle of the period. Now, some one might prefer to conclude that we ought not to hear the tone 25 all the time, but at first a tone somewhat lower than this, gradually rising slightly and falling again in pitch towards the end of the period. Whether we should draw this conclusion I will not attempt to decide. Neither do I care to express a definite opinion as to what we actually hear. Let the reader who wants to know this find it out by an experiment of his own. What I must point out, however, is the fact that the time interval between two maxima is not necessarily the time between two stimulations. In a provisional way, the interval between two maxima or between two minima or between two points of inflection or between two points of any other name and definition may be used thus, but let us always remember that this is only a provisional, an artificially simplified method, which can scarcely yield more than a rough approximation of what actually happens.

Another section of the partition moves up and down twenty-four times during the period. The length of this section, which determines the relative intensity of **Do we hear the tone 24?** the tone heard, is derived from the table as being twenty-one ($221 - 200$). If we look at the time interval between the successive maxima, we find this to be at the beginning of the period one hundred and forty-seven, to decrease gradually to one hundred and forty-five at the maximum twenty-three, to be two hundred and thirty from maximum twenty-three to maximum twenty-seven (maximum twenty-five has disappeared, as seen in figure 21), and to fall again to one hundred and forty-five. Here again, I will not attempt to decide what we ought to expect theoretically, because we have no right to deduce anything definite from a theory in a direction in which this theory is as yet professedly indefinite, in which it obvious-

ly lacks as yet all details, owing to the deficiency of the requisite experimental data. I can only repeat here what I said in the preceding paragraph.

Before we continue this attempt at an interpretation of figure 21, let us consider an imaginary case the application

No indiscrimi- nate counting of stimuli allowed

of which to our figure will soon make itself clear. Imagine that during half a second a nerve end receives in regular intervals fifty stimulations, but during the following half-second no stimulations at

all; then again for half a second fifty stimulations in regular intervals, and again for half a second none; and so on. What could we hear in such a case, but a tone for half a second, nothing for half a second, a tone again for half a second, nothing again for half a second, and so on. And what tone would it be? Plainly the tone which we ordinarily call 100, because the frequency with which fifty stimuli are received in half a second is the same as that with which one hundred are received in one second. I need not waste any effort in trying to prove what is self evident, namely that it would be absurd to count in a case like this simply the number of stimuli during any whole second and to expect, these being fifty, that we should hear the tone 50. And yet this way of counting has been actually proposed. But this proposition may well be ignored.

Now let us return to the interpretation of figure 21. The third section of the partition, the length of which is twenty-

What beats do we hear?

four (191 — 167), receives stimulations in approximately equal intervals until about the maximum twenty-three when there is no stimulus at all until about the maximum

twenty-nine. With the rough approximation here possible we may say that there is no stimulus during about one-tenth of the period. From our discussion in the preceding paragraph it follows that during about nine-tenths of the period we should

hear a tone and during one-tenth of the period we should hear nothing so far as the nerve ends of this section are concerned. The pitch of the tone we must expect to lie between the tones 24 and 25, acording to the probable frequency with which the stimulations are received during that part of the period during which they are received.

It is plain that the fourth, fifth and following sections of the partition must move up and down very much the same as the third section does, with this difference only, **The "mean" tone** that for each further section the pause when no stimulations at all are received becomes longer and longer. The total sensation, then, which is derived from the sum of the nerve ends of the third and the following sections must be a tone of a certain intensity at a certain time when all these sections mediate the sensation, but becoming weaker and weaker as one after another of the sections stops moving until for a moment it ceases altogether, then appearing again and increasing up to its former intensity. And so on again and again. That is to say, we hear this tone "beating." And since its pitch lies probably somewhere between 24 and 25, between the "primary" tones (perhaps its pitch is not quite constant but may vary slightly during each period), I propose to call it the "mean tone" (German: Zwischenton). The question whether we hear such a mean tone I do not care to answer here, this discussion being devoted to theory, not to experimental research. Let the reader who desires make observations of this kind himself.

The farthest section of the partition set in motion by this movement of the stirrup moves up and down only once during the period. The nerve ends located **The difference** here receive one shock during each period **tone** and convey therefore the sensation of the tone 1, the difference tone of this case. The intensity of the difference tone, corresponding to the length of this section of the partition, is two.

It is not impossible, however, it is even probable, also that a few of the sections just preceding this last convey the sensation of this difference tone, instead of that of the mean tone. The last section which may convey the sensation of the mean tone moves only twice up and down during the period, in quick succession. This double movement is followed by a long pause during which no movement occurs. Now, experimental research of recent years has proved that two shocks received by the auditory nerve ends may be sufficient to give the sensation of the tone corresponding to the frequency with which the two shocks are received—but only within the middle region of the tonal series. Towards either end of this series four, six, and even more shocks are found to be necessary for the sensation of the tone corresponding to the frequency of the shocks. What, then, will be the consequence of choosing the tones 24 and 25 somewhat higher? The section of the partition which makes the two up and down movements in quick succession can no longer convey the sensation of a short mean tone. If there is only one period of movement, no sensation at all will then result. But if many periods succeed, it is much more likely that the double movement of the partition section will have the effect of a single shock than no effect at all; and the repetition of this shock in each succeeding period must result in the sensation of the tone 1, the difference tone.

If the tones 24 and 25 are chosen still higher, it becomes improbable that even three shocks received by the nerve ends in quick succession between two long pauses can give the sensation of a short mean tone. In this case it is highly probable also that the second section before the last conveys the sensation of the difference tone. And so a few more of those more distant sections may convey the sensation of the difference tone instead of the mean tone.

If the difference tone results exclusively from the function of the nerve ends located on the last moving section of the partition, its relative intensity is two, according to the above table. But if the difference tone results from the function of the nerve ends of further sections, its relative intensity must be higher and the maximum intensity of the mean tone correspondingly lower. That is, the phenomenon of a beating mean tone must be the less pronounced the more audible the difference tone; and the difference tone of a small interval like the one in question must be the more audible the higher the pair of primary tones in the tonal series.

Summarizing now our interpretations of figure 21, we must say that so far as the meager data reach from which we can draw theoretical conclusions, the following seems likely to be the total impression (listening with one ear, having the other ear plugged): 1. A tone 25 of the constant, but comparatively weak intensity nine; 2. a tone 24 of the constant, but comparatively weak intensity twenty-one; 3. a mean tone (perhaps slightly varying in pitch during each period) of an intensity which varies once during each period from zero to a definite maximum intensity and back to zero. This maximum intensity may be (under the most favorable conditions) as high as (relatively) three hundred and sixty-eight, but must be much less if the primary tones are above the middle region of the tonal series. Its being less means that the "beats" are less pronounced; 4. a difference tone the relative intensity of which may be (under the most unfavorable conditions) as low as two. Its intensity, however, may be greatly increased, at the expense of the maximum intensity of the beating mean tone, in case the pitch of the primary tones is raised.

The combination 24 and 25; summary

Before we take up the theoretical discussion of further tone combinations, the reader ought to obtain some information concerning the difference tones which we hear in addition to the "objective" tones in the several combinations. To give such information of this kind as is indispensable, I shall state here the laws of these phenomena in as clear and comprehensible a manner as possible. These laws given below do not pretend to tell all the difference tones which we might possibly hear in every possible combination of objective tones. Neither do they tell the relative intensities of the difference tones, although this is a matter of no small importance. Laws of difference tones of this scientific perfection are as yet not known and may never be known. The laws below merely tell those difference tones which one is most likely to hear in those combinations which correspond to relatively simple ratios of the vibration rates and are therefore (musically and otherwise) particularly interesting. These laws are the following four:

Laws of difference tones

In case the ratio of the vibration rates does not differ much from $1:1$, let us say $11:12$, or $9911:9989$, a single difference tone is audible, whose pitch corresponds to the pitch of a tuning fork the vibration rate of which is equal to the difference of the vibration rates of our case. In addition to the difference tone, however, beats are usually clearly audible, and a mean tone may be audible too which lies between the two primary tones. If the interval is quite small, this mean tone is usually more pronounced than either of the primary tones, particularly when we hear with one ear only, having the other ear plugged. The beats just mentioned seem to be the fluctuations of the intensity of the mean tone rather than of the primary tones, if we use one ear only.

First law of difference tones

A second class of ratios which is of particular interest, is that of the ratios whose numbers differ by one. In each

**Second law of
difference tones**

of these cases the difference tone 1 is audible, but often quite a number of additional difference tones can be perceived. If the numbers of the ratio are rather small, as in the case of 5 : 4, all the tones from the highest, that is, 5, down to 1 are without any great difficulty noticeable. As we study ratios of increasing numbers, the tones following directly upon 1 (in a rising direction) seem to have a tendency to drop out. And if we go on in the same

Objective tones	Difference tones easily audible
2, 1	–
3, 2	1
4, 3	2, 1
5, 4	3, 2, 1
6, 5	4, 3, ?, 1
7, 6	5, 4, ?, 1
8, 7	6, 5, ?, 1
9, 8	7, 6, 5, ?, 1
10, 9	?, 1

way, we soon find only one difference tone left, the tone 1. We have then simply reached a case in which the difference tone is determined by the first law above. The accompanying table represents this class of ratios with their difference tones.

A third class of ratios are the ratios made up of comparatively small numbers, representing intervals less than an

Third law of difference tones

octave. In these cases three difference tones are often easily noticeable, one corresponding to the direct difference of the vibration rates $(h - l)$; one corresponding to the difference between the latter number $(h - l)$ and the vibration rate l of the lower primary tone, that is, $(2l - h)$; and one corresponding to the difference between the just mentioned differences $(h - l)$ and $(2l - h)$, that is $(2h - 3l)$. It is to be noticed, however, that a difference tone is rarely audible which corresponds to a difference larger than the subtrahend; for example, the primary tones 9 and 5 produce the difference tones 4 and 1, but not $3 = 4 - 1$, or at least not an easily noticeable tone 3, three being larger than one. The following table contains a few examples of this class:

Objective tones	Difference tones easily audible
8, 5	3, 2, 1
5, 3	2, 1
9, 5	4, 1
7, 4	3, 1
11, 7	4, 3, 1

The fourth class are the ratios made up of comparatively small numbers, representing intervals larger than an octave.

Fourth law of difference tones

The first fact to be noticed here is the lack of an easily observable difference tone corresponding to the direct difference of the two vibration rates. Such a tone, if audible, would lie between the primary tones. As a rule, only one difference tone is easily noticeable in these cases, which can be found according to the following

rule: Find the smallest difference between the larger num-
ber of the ratio and any multiple of the smaller number. The
table contains a few instances of this class:

Objective tones	Difference tones easily audible
11, 4	$1 = 3 \times 4 - 11$
12, 5	$2 = 12 - 2 \times 5$
9, 4	$1 = 9 - 2 \times 4$
11, 3	$1 = 4 \times 3 - 11$
5, 2	$1 = 5 - 2 \times 2$
8, 3	$1 = 3 \times 3 - 8$

Let me repeat that the above rules do not pretend to
represent scientific laws in the strict sense of the word. They
are stated here chiefly for a practical pur-
The use of　　pose. If the reader who is unfamiliar with
such laws　　difference tones will use the above "laws"
as directions for observation and obtain
a first hand knowledge of the phenomena of difference tones,
he will be more interested in the theoretical discussions which
are to follow, and able to decide for himself in what di-
rections the mechanical theory is yet most undeveloped and
most wanting in details.

Let us apply our theory now to the combination of two
sinusoids of the relative periods nine and four, that is, of the
relative frequencies 4 and 9. The com-
The combination　pound curve, representing the function
4 and 9　　　　　$$f(x) = 1.99 + \sin 4x + \sin 9x$$
is shown in figure 22. The period is made
to begin and to end with the lowest ordinate value of the
function, zero, because this has certain technical advantages.

It is, of course, in a periodical function, entirely irrelevant for the mechanical theory what point we regard as the beginning of the period. The accompanying table contains the pairs of corresponding coordinate values of all the maxima and minima of the curve. These values are found by computing a large number of pairs of values and selecting from them

INTERVAL 4:9, EQUAL AMPLITUDES

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Max.	+ 169	119	368	P	338
Min.	— 16	318	183	Q	185
Max.	+ 75	471	274	R	91
Min.	— 199	696	0	A	274
Max.	+ 110	929	309	B	309
Min.	— 2	1094	197	C	112
Max.	+ 142	1275	341	D	144
Min.	— 189	1512	10	E	331
Max.	+ 42	1724	241	F	231
Min	— 42	1876	157	G	84
Max.	+ 189	2088	388	H	231
Min.	— 142	2325	57	I	331
Max.	+ 2	2506	201	J	144
Min.	— 110	2671	89	K	112
Max.	+ 199	2904	398	L	309
Min.	— 75	3129	124	M	274
Max.	+ 16	3282	215	N	91
Min.	— 169	3481	30	O	185
Max.	+ 169	3719	368	P	338

those which have the highest and lowest ordinate values. This computation is a very slow process, but has no limit of accuracy. Figure 23 shows the positions of the partition belonging to the maxima and minima of figure 22. We see that at A the initial forty sections of the partition are in their

upper positions. At B, the first thirty-one of them are at their lower limits. At C, the stirrup has caused eleven sections to assume their upper limits. From C to D, the stirrup moves inwards through fourteen units of space, pushing down the eleven sections which were up at C, leaving the following twenty unmoved since they are down already, and pushing down three more, so that now the first thirty-four
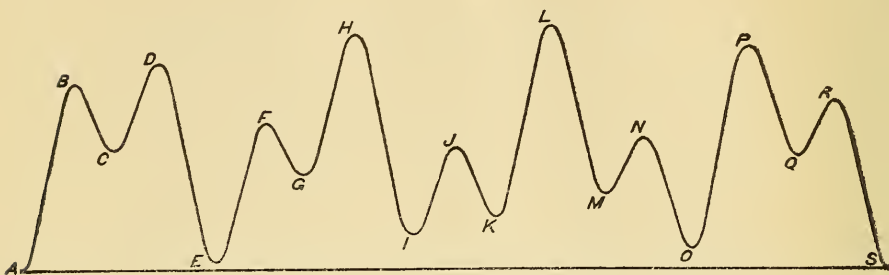
Fig. 22. The combination 4 and 9

sections of the partition are down, six further sections are up, and all the following ones are in their normal positions. From D to E the stirrup makes an outward movement through thirty-three units of space, moving up the first thirty-three sections of the partition. From E to F, the stirrup moves inwards through twenty-three units of space; and so on. At S, we find the partition in the same position as at A, our starting point; then, a new period begins.

Let us now try to interpret the figure. We can easily see that the first eight sections move down and up again nine times during the period. This would mean **Do we hear 9?** that the nerve ends located on this section convey to our mind the sensation of the tone 9 of the relative intensity eight. The ninth section of the partition moves down and up only eight times during the period; but after our discussion about the omission of stimuli

it is clear that we should not be justified in concluding that we must hear the tone 8. This tone would be audible only if the frequency with which the stimuli occur on the ninth section was less than the frequency on the first eight sections. However, there is no reason why we should regard the fre-
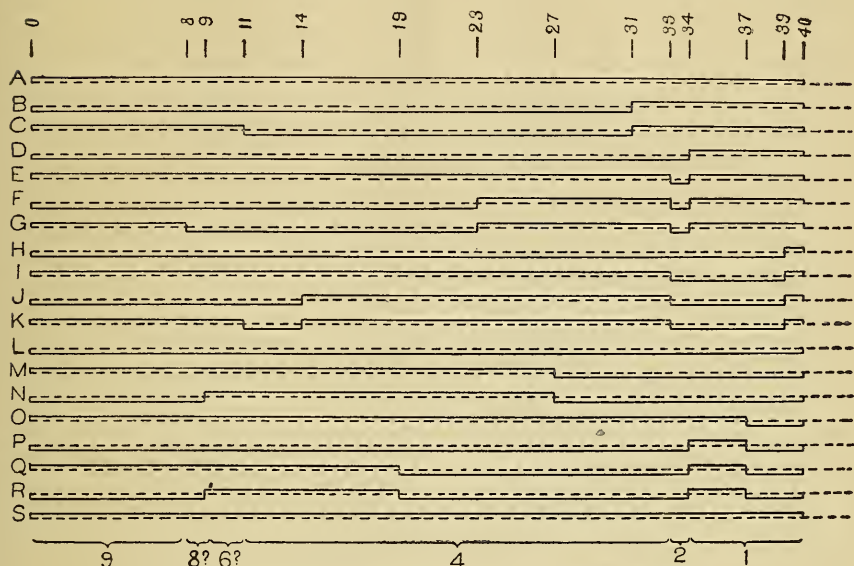
Fig. 23. The combination 4 and 9. Compare figure 22

quency as different. It seems most probable, then, that the nerve ends of the ninth section convey to us the sensation of the tone 9, but with a short pause (or possibly, because of the after-sensation, a diminution of intensity only) at the moment about G, when no stimulation takes place. Our total impression of the tone 9 is, of course, the sum of the sensations conveyed by all the nine initial sections. This means that the tone intensity perceived would, on the whole, be nine; but that for one moment in each period this intensity of the tone might suddenly be slightly decreased. It does not

seem improbable—so far as our theoretical data permit us to draw a conclusion—that such a sudden, but weak decrease in intensity might become noticeable as a kind of just perceptible "beat." I leave it to the reader to decide experimentally whether the tone 9 in this combination appears slightly "rough" or perfectly "smooth."

The tenth and eleventh sections of the partition move down and up six times during the period. But we must remember here from our previous discussion that—in order to conclude as to the tones to be heard—no indiscriminate counting is permissible. Mere counting of stimuli would indicate the tone heard only in case it seems probable that these stimuli occur in equal or approximately equal intervals. Now, a survey of figure 23 does not make it appear probable that the stimuli on the two sections in question occur in even approximately equal intervals. The partition moves down at F and remains in the lower position until it moves up at I. It moves down at J and immediately, at K, up again. Down at L and up at M. In this upper position it remains until P, when it moves down. At Q it is up again, to stay in the upper position until B, when it moves down. At C it is up again. At D it moves down, at E up, and at F down again. Are we justified in concluding that the nerve ends located on these two sections of the partition must convey to our mind the sensation of the tone 6 of the intensity two; or any other definite sensation? I do not know how to answer this question. If we knew the time intervals between the successive stimuli exactly, we might attempt to decide whether one or the other sensation would be more or less probable in this case. But we know that figure 23 is only an approximate, not an exact representation of the actual movement of the partition. It is a certain comfort in this dilemma that the prac-

Do we hear 6?

tical importance of a decision in this case is rather small, for the reason that, whatever sensation these two sections might produce, it would be a sensation of the relative intensity two only, a rather weak sensation compared with the tones which appear theoretically certain.

The twelfth, thirteenth, and fourteenth sections of the partition move down at B, a second time at F, a third time at J, and a fourth time at P. These sections, therefore, move down and up four times during the period in approximately equal intervals. The five following sections of the partition move down at B, a second time at F, a third time at L, and a fourth time at P. These sections, therefore, move down and up four times during the period in approximately equal intervals. The four sections from the twentieth to the twenty-third move down at B, a second time at F, a third time at L, and a fourth time at P. These sections, therefore, move down and up four times during the period in approximately equal intervals. The following four sections move down at B, a second time at H, a third time at L, and a fourth time at P; that is, four times during the period in approximately equal intervals. The four sections from the twenty-eighth to the thirty-first move down at B, a second time at H, a third time at L, and a fourth time at P; again, four times during the period in approximately equal intervals. The thirty-second and thirty-third sections move down at D, a second time at H, a third time at L, and a fourth time at P. These sections, therefore, move down and up four times during the period in approximately equal intervals. It follows that according to our theory we must expect to hear the tone 4 of a relative intensity twenty-two, since it is produced by all the sections from the twelfth to the thirty-third.

Do we hear 4?

The thirty-fourth section of the partition moves down at D, and up again at O; down at P, and up again at S. That is, the nerve ends of this section receive

Do we hear any difference tones? two stimuli during the period. We may expect to hear, therefore, the tone 2 of the relative intensity one. The three follow-

ing sections of the partition move down at H and up again at O. The nerve ends on these sections receive, therefore, one stimulus during the period. The next two sections move down at H and up again at S. The nerve ends here receive one stimulus during the period. The fortieth section moves down at L and up again at S. The nerve ends here receive one stimulus during the period. We must hear, then, the tone 1 of the relative intensity six. The tones 2 (weak) and 1 (strong) are the only difference tones in this case which we can derive from our theory with some degree of certainty.

Summarizing now the results derived from our representation of the movements of the partition in the case of the

The relative intensities compared ratio 4:9, we find that we must expect to hear the tones 9, 4, 2 and 1, with the relative intensities nine, twenty-two, one and six; leaving out of discussion the doubt-ful sensation of the intensity two which

may be conveyed to our mind by the tenth and eleventh sections. Now, it is quite natural to ask the question whether we hear these tones with just these relative intensities. Un-fortunately, no exact answer to this question is possible, because this matter, owing to technical difficulties and other circumstances, has never been experimentally subjected to accurate measurement. It is known, however—what also ap-pears in the above statement of our results—that in a com-bination of two tones the higher one loses in intensity, com-pared with the lower one. Yet it is doubtful if this loss in intensity is so great as the number nine indicates, compared

with twenty-two. The present writer at least is inclined to doubt this. He believes that the theory, representing only an approximation to what actually happens in the organ of hearing, exaggerates the degree of this loss of intensity on the part of the higher tone. He is also inclined to believe that the theory exaggerates the relative intensity of the difference tone 1, which was found to be six. In reality, this tone seems to be somewhat weaker than is indicated by this number.

Let us remember, now, the provisional assumptions which we made in order to render the graphic representation of the movement of the partition as simple as possible. We may raise this question: Is not, perhaps, the above disagreement between theory and experimental observation a result of one or more of these provisional assumptions? I shall demonstrate that this is indeed the case. Or, more exactly, I shall demonstrate that, if we omit one of these assumptions and take into account in its stead the actual anatomical conditions so far as these are known, we change the results of the theory in such a direction as to diminish the exaggerated loss of intensity of the higher primary tone and also the exaggerated intensity of the difference tone.

The third and fourth provisional assumptions recalled

The partition was provisionally assumed to be of equal width all along the tube. As a matter of fact, its width near the windows is only one-twelfth or one-tenth (measurements differ somewhat) of what it is at the far end of the tube. And further, it is to be noted that the width of the partition does not increase uniformly along the tube, like the area between the dotted lines of figure 24, but that it increases first rather rapidly, later more slowly, like the area between the curved lines. The figure, however, does not represent the true relation between the width and

The partition is narrower near the windows

the length of the partition. The partition as a whole is much narrower in comparison to its length than appears in the figure. Let us try, then, to get a clear conception of the functional significance of these facts. It is of no particular importance, in this connection, whether the measurements upon
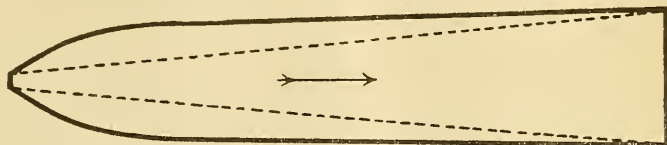

Fig. 24.   Shape of the partition

which the following considerations are based are more or less incorrect, as they probably are; for our intention is merely to get an idea of the general direction in which the actual shape of the partition changes the results of a theory having provisionally assumed that the partition is everywhere of equal width.

When the partition yields in either direction, up or down, its former place is taken by the fluid of the tube. Let us call the quantity of fluid which has taken positions formerly occupied by the partition "the displaced fluid." Now, it is plain that the quantity of displaced fluid must always be approximately proportional to the distance through which the stirrup has moved since its last reversal of movement. If the partition were equally wide everywhere, then any section of equal length, far from or near the windows, would make room, in moving from one limit to the other, to the same quantity of displaced fluid as any other section. And then, plainly, the length of that part of the partition which is caused to move from one limit to the other would always be proportional to that part of the stirrup movement which caused it to move.

A unit of stirrup movement equals a unit of displaced fluid

This is the effect of our provisional assumption. But if the partition tapers as it does, a unit of displaced fluid (corresponding to a unit of stirrup movement) is made room for by sections of the partition of very unequal length according as the displaced fluid unit is located nearer or farther from the windows. Where the partition is narrow, a longer section would have to move in order to make room for a unit of displaced fluid. Where the partition is wider, a shorter section would make room for the same quantity of fluid.

Since, then, tone intensity depends on the length of the partition section which is jerked up and down, and since this length is not proportional to the given value of the stirrup movement, it is useful to have a table showing the partition lengths corresponding to various stirrup movements in order to get a clear idea of the influence of the tapering of the partition upon the relative tone intensities. To simplify the computation of such a table, it is well to restrict it to a short distance from the windows, so that we

The computation of a table
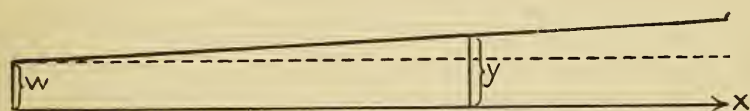
Fig. 25. The partition widens

may approximately assume the partition to increase uniformly in width within this distance. Let us call w the smallest width of the partition, near the windows; let us assume that a distance from the windows equal to $50w$ the width of the partition is $6w$, and let us assume a uniform increase of width. Let us call y the width at any point of the partition and x the distance of this point from the beginning

near the windows. We then know (Fig. 25) that the ratio
$\frac{y-w}{x}$ is equal to the ratio of $\frac{6w-w}{50w}$

$$\frac{y-w}{x} = \frac{6w-w}{50w} = \frac{1}{10}$$

$$y = w + \frac{x}{10} = \frac{10w+x}{10}$$

The area described by the cross-section of the partition in
being jerked from one limit to the other may be called a at
the point where the width of the partition
is smallest, α at any arbitrary point of the
The area
described by a
cross-section of
the partition
partition. These areas, let us assume, are
geometrically similar. This assumption
possesses a higher degree of probability
than what would follow for the areas from
the third provisional assumption made above for the sake of
simplicity. It then follows that the ratio of the areas is equal
to the ratio of the squares of the widths of the partition at the
same points.

$$\frac{\alpha}{a} = \frac{y^2}{w^2}$$

$$\alpha = \frac{a\,y^2}{w^2}$$

For y we substitute its value found above and have then the
equation:

$$\alpha = \frac{a(10w+x)^2}{10^2w^2}$$

The left side of this equation is a measure of the area
described by the cross-section of the partition at the point
x, in being jerked from one limit to the other. The right
side of the equation contains the variable x, the distance of
any point of the partition from its beginning near the windows,

and the two constants a and w. The former of these constants is the area described by the initial point of the partition in moving from one limit to the other, of whatever form this area may actually be found to be. The latter is the width of the partition at the initial point.

The mathematical reader immediately sees that that quantity F of displaced fluid for which room is made by a move-

The quantity of fluid for which room is made
ment of any given section of the partition is determined by the following equation, which can be easily integrated.

$$F = \int_{x_1}^{x_2} a\, dx$$

In order to integrate this equation we have to express a as a function of x. This has been done above under the temporary assumption of a uniform increase of width. The result is stated in the equation just preceding the last. We then have

$$F = \int_{x_1}^{x_2} \frac{a}{100w^2}(10w+x)^2 dx =$$
$$= \frac{a}{300w^2}\left[(10w+x_2)^3 - (10w+x_1)^3\right],$$

where x_2 is the farther, x_1 the nearer of the two points enclosing whatever section of the partition is in question.

If the section in question is an initial section of the partition, then x_1 is equal to zero, and the quantity of displaced fluid is

$$F = \frac{a}{300w^2}\left[(10w+x_2)^3 - (10w)^3\right]$$

Let us regard the partition as consisting of sections each of the length of w. We can find, then, the quantities of displaced fluid for which room is made by the first section, the first two, the first three, the first four, and so forth, sections

by making x_2 successively equal to w, to $2w$, to $3w$, to $4w$, and so forth.   If $x = nw$, we have

$$F = \frac{a}{300w^2} \left[(10w + nw)^3 - (10w)^3 \right] =$$
$$= \frac{a\,w}{300} \left[(10+n)^3 - 10^3 \right].$$

Let us arbitrarily regard $\dfrac{a\,w}{300}$ as the unit of displaced fluid.   We could then easily compute a table which contains the number of fluid units displaced by the number n of partition units.   If the number of partition units is, for example, three, the quantity of displaced fluid is

**Two tables
possible**

$(13^3 - 10^3)$ units and so on.

More useful, however, is a table which progresses in a regular series of units of fluid and tells us—in decimals—the lengths of the initial sections which make room for these quantities of fluid; for our representation of the movement of the partition tells us the quantities of displaced fluid, and the corresponding section lengths are to be found in order to obtain a more correct idea of the relative tone intensities.   In order to compute such a table it is advantageous to use a larger fluid unit than the above.   Let us determine the total quantity of fluid for which room is made by the partition section from $x = 0$ to $x = 50w$, that is, the whole part of the partition near the windows for which we have assumed a uniform tapering or change of width; and let us—arbitrarily—regard one-fiftieth of this quantity as the fluid unit.

$$F = \frac{a\,w}{300} \left[(10+50)^3 - 10^3 \right] =$$
$$= \frac{215000\,a\,w}{300}$$

The fluid unit, defined as one-fiftieth of the above quantity, is therefore

$$\frac{4300\ a\ w}{300}$$

Any number m of such fluid units is then

$$\frac{4300\ a\ w\ m}{300}$$

We derived above the following equation between fluid quantities and partition lengths

$$F = \frac{a}{300\ w^2} \left[(10w + x_2)^3 - (10w)^3 \right]$$

In this equation we have to substitute for F the above expression of fluid quantity and then to solve the equation for x_2.

$$\frac{4300\ a\ w\ m}{300} = \frac{a}{300w^2} \left[(10w + x_2)^3 - (10w)^3 \right]$$

$$(10w + x_2)^3 = 10^3 w^3 + 4300 w^3 m$$

$$10w + x_2 = w\sqrt[3]{1000 + 4300m}$$

$$x_2 = (\sqrt[3]{1000 + 4300m} - 10)w$$

The following table contains the corresponding values of m and x_2, measured in the unit of length w.

Let us see, now, how we must use the table of fluid quantities and partition lengths. We recall that any unit of stirrup movement causes the displacement of a unit of fluid. What we have called above "the relative intensities of the tones heard" refers directly to relative numbers of units of stirrup movement; indirectly also to relative numbers of units of displaced fluid, since it is highly probable that the quantity of displaced fluid is approximately proportional to the extent of a stirrup movement. What we want

The use of the table

TABLE OF THE RELATIONS BETWEEN FLUID DISPLACEMENT AND
PARTITION LENGTH

m	x	m	x	m	x	m	x	m	x
1	7.43	11	26.42	21	35.03	31	41.22	41	46.18
2	11.25	12	27.47	22	35.73	32	41.75	42	46.62
3	14.04	13	28.46	23	36.40	33	42.28	43	47.07
4	16.30	14	29.41	24	37.05	34	42.80	44	47.50
5	18.23	15	30.31	25	37.70	35	43.31	45	47.94
6	19.92	16	31.17	26	38.32	36	43.81	46	48.36
7	21.45	17	32.00	27	38.92	37	44.30	47	48.78
8	22.83	18	32.80	28	39.51	38	44.78	48	49.19
9	24.11	19	33.57	29	40.10	39	45.26	49	49.60
10	25.30	20	34.31	30	40.65	40	45.72	50	50.00

to know now, is the length of the several sections of the partition of which—in the last case of tone combination, 4 and 9—the first or initial one moves up and down nine times and produces the tone 9, the second produces no definite tone with

certainty, the third produces the tone 4, the fourth the tone
2, and the fifth the tone 1.

The fluid quantity for the tone 9 is measured, as we found
above, by the relative number nine. Now, let us, for exam-
ple, assume that this means an equal number of fluid
units in our table. We then read off the corresponding par-
tition length as being 24.11 units. The fluid quantity for the
uncertain tone was measured as two units. But now, we can-
not simply read off from the table the number of partition
units corresponding to two; for the partition section making

TONE INTENSITIES IN THE COMBINATION 4 AND 9

Tones	Uniform width	Tapering
9	22.5%	52.7%
Uncertain	5.0%	5.0%
4	55.0%	34.8%
2	2.5%	1.1%
1	15.0%	6.4%

room for these two fluid units is not an initial section. We
must read off, therefore, the value corresponding to eleven
fluid units (26.42) and subtract from this the value correspond-
ing to nine fluid units (24.11). We thus see that the length
of the partition section about the tone of which we could not
come to a decision is 2.31 units. The fluid quantity for the
tone 4 was measured as twenty-two. But here again we can-
not simply read off the length of the partition section pro-
ducing this tone, because this section is not an initial section.
We must read off the values for $9+2+22=33$ and for $9+2=11$
and subtract the latter from the former. These values are
42.28 and 26.42. The length of that section of the partition

which moves up and down four times is therefore 15.86 units.
The intensity of the tone 2 is one fluid unit. The length of
the partition section corresponding to this fluid unit is 42.80—
42.28=.52. The fluid quantity for the tone 1 is six fluid units.
We have to read off from the table the values corresponding
to 9+2+22+1+6=40 and to 9+2+22+1=34 fluid units. These
values are 45.72 and 42.80. The length of that section of the
partition which produces the tone 1 is therefore 2.92 units of
the partition.

The relative intensities of the four tones 9, 4, 2, and 1,
would then be, not as nine to twenty-two to one to six, but as
24.1 to 15.9 to .5 to 2.9 ; and the tone about
which we could not reach a definite con-
clusion would have the relative intensity
2.3 instead of two. For the sake of better
comparison let us express the relative in-
tensities in percentages. The table shows
in one column the tone intensities in case we regard the par-
tition as of uniform width and in another column the intensi-
ties in case we regard the partition as tapering and possess-
ing those properties upon which the present computation is
based.

**The relative
intensities of the
tones 9, 4, 2,
and 1**

We must not, of course, regard the result found in the
second column of intensities as any more final than that in
the first column. We have assumed that
the initial section of the partition tapers
uniformly so that, the initial width being
w, its width is $6w$ at a distance of $50w$. But
we do not know that it tapers just this way. We have further
assumed that the areas described by cross-sections of the par-
tition in moving from one limit of position to the other, are
geometrically similar. But we do not know whether they are
or not. We have further assumed that the total movement of
the partition in this case extends just to the distance of $45.72w$.

**This result
not final**

But this is an arbitrary assumption, and the results of the table, as is shown farther below, would look different if the total movement did not extend just so far, but farther or less far. We must not, then, regard this result as final, but simply observe if it tends to change the relative intensities in such a direction as might correct the intensities which seemed somewhat objectionable. Now, we objected, first, to the fact that the higher of the primary tones had such a slight intensity compared with the lower one, 22.5 per cent compared with 55.0 per cent. Now we see that taking into account the tapering of the partition raises the intensity of the tone 9 to 52.7 per cent and lowers that of the tone 4 to 34.8 per cent. As stated before, these particular figures must not be regarded as a final result. It is irrelevant that now the lower tone is weaker than the higher. What is important is the fact that the influence in question is in the direction in which it must be in order to correct the objectionable features of the former computation. A further result of this influence is the reduction of the former intensity of the difference tone 1, which we regarded as rather high, from 15.0 per cent to 6.4 per cent—again a change in the desired direction.

We can obtain here a more special insight in addition to the general insight into the fact that tapering of the partition tends to increase the intensities of the tones produced by the initial sections, to decrease the intensities of the tones produced by more distant sections of the partition. More especially, we shall observe that the amount of this increasing or decreasing influence varies according as the total length of the partition section set in motion varies, that is, as the total intensity of the compound sound heard varies. Imagine, for example, three tones, which we call A, B, and C, being produced by successive sections of

The relative intensities not independent of the absolute intensity of the compound sound

the partition.   Imagine further that the quantity of displaced
fluid for the tone A is 20 per cent of the total amount of fluid
displaced by the compound sound wave, that the quantity
for B is 50 per cent, and the quantity for C 30 per cent.   This
is a percentage which might easily be found in an actual case.
The pitch of the tones A, B, and C is irrelevant.   The table
below contains all the values which are of interest to us, for
two cases.   In the first case the actual   fluid quantities are
two, five, and three, by assumption; in the second case they
are ten, twenty-five, and fifteen.   That is, the stirrup move-
ment in the second case is of the same form, but exactly five
times as large as in the first.

Quantities of dis-placed fluid			Length of sections (absolute values)				Length of sections (percentages)		
A	B	C	A	B	C	Σ	A	B	C
2	5	3	11.3	10.2	3.8	25.3	44.7%	40.3%	15.0%
10	25	15	25.3	18.0	6.7	50.0	50.6%	36.0%	13.4%

The table shows that the tone intensities do not increase
proportionally to the increase in the amplitude of stirrup move-
ment.   The amplitude in the second case is five times that of
the first case; but the total intensity (Σ) of the audible sound
in the second case is less than twice that of the first case (50.0
compared with 25.3).   The table shows further that the inten-
sity of the tone A is in the first case 44.7 per cent, in the second
case 50.6 per cent.   That is the increase in the intensity of the
whole sound is favorable to the relative intensity of the tone
produced by the initial section of the partition. The percentage
intensity of this tone, A, is increased at the cost of the tones
B and C, the percentages of both of which are diminished.

Thus far we have studied the effect upon the relative tone intensities of initial and more distant sections which would

Increase in width of partition not uniform
result from a uniform increase in width of the partition as compared with a uniform width. But we know that the partition does not increase uniformly, but rapidly at first, near the windows, and more slowly the farther we go from the windows (Fig. 24). To understand the theoretical result of this manner of increase, it is not necessary to compute a new table. It is plain that, if a more distant section increases less than we assumed in computing the preceding table, showing the corresponding values of m and x, this would cause a longer piece of this distant part of the partition to move in order to make room for a certain quantity of displaced fluid. That is, the decrease in the broadening of the partition would counteract the effect last discussed. We saw in the preceding paragraph that an increase in the intensity of the whole sound does not leave the relative intensities of the partial tones unaltered, but favors the intensities of the tones on the initial sections, reduces those on the distant sections. But now, if we increase the intensity of the whole sound, we throw the tones of the more distant sections on still more distant sections, that is, on sections where the broadening of the partition is much less than that assumed in the table. Consequently, the tones of distant sections cannot lose in percentage as much as a derivation from the table would indicate, but might even gain somewhat in percentage of intensity through an increase of the total intensity of the sound.

The preceding paragraphs must impress us with the perplexity of our situation. We want to comprehend the facts of audition as depending on the structure and function of the sense organ. But every endeavor to enter into the details of the function of the organ is thwarted by the poverty and inaccuracy of our anatomical knowledge. We cannot obtain a definite idea of the intensities of the various physiological processes resulting from a compound aerial wave unless we know exactly the manner of increase in width of the partition. It is not sufficient to know that it increases first rapidly, then slowly. We need a very exact measurement of the width of succeeding cross-sections of the partition and of the distance of each of them from the beginning of the partition near the windows.

The need of a more accurate and detailed anatomical knowledge

On the other hand, we need also a much more detailed and accurate comparison of the relative intensities of the components of stronger and weaker compound sounds, based on psychological experimentation and observation. Thus far, practically nothing in this regard is known with exactness. It is to be hoped that, in spite of the extraordinary technical difficulties and the costliness of the apparatus required for such investigations, an accurate knowledge of these psychological facts will be obtained. We need this knowledge because some of the constants contained in the mechanical theory may never become directly measurable, for example, the elastic properties of the partition, and, therefore, will have to be inferred from their psychological consequences.

The need of a more accurate observation of the psychological facts of hearing

Two consequences of the particular shape of the partition which we have just discussed in as much detail as anatomical knowledge permits should be emphasized. The first of these is of the greatest biological significance. It is certainly important for the animal to be very sensitive to sound, that is, to be able to hear sounds which are very weak and cause only a minute movement of the stirrup. Now, the initial part of the partition being exceedingly narrow, even the minutest quantity of fluid displaced by the stirrup must spread considerably lengthwise over the partition and thus stimulate quite a number of nerve ends. But it would not be advantageous to have the partition equally narrow all along. In that case comparatively weak objective sounds would cause the whole partition to move up and down and the displaced fluid for which no room can be made by the partition, to flow back and forth through the "safety valve." Strong objective sounds would then make the same impression upon the animal as sounds of medium physical intensity. This disadvantage is overcome by the partition's tapering, by its being narrow at the beginning, but wide farther on, so that even sounds of considerable strength do not involve the whole partition. But again, there would be a disadvantage if the partition's width increased uniformly; for then the relative intensities of simultaneous tones —as we have seen—would not be even approximately independent of the absolute intensity of the total sound. This disadvantage might be avoided by the width increasing first rapidly, then more and more slowly. If it is thus avoided, either partially or totally, we do not exactly know because of lack of exact anatomical data.

Two important consequences of the partition's shape. Sensitiveness of the ear

The second of the consequences to be emphasized is probably of little biological significance, but possibly of some

Conditions more or less favorable to the observation of difference tones

importance to the student observing difference tones in a psychological laboratory. It is quite possible that, as a result of the tapering not being uniform but decreasing as the windows are left behind, the relative intensity of difference tones, which are obviously produced by the more distant

sections of the partition, is somewhat greater when the absolute intensity of the whole sound is rather great. If this is so, it would be advisable to use for the observation of difference tones fairly strong primary tones rather than weak ones. Whether this conclusion is borne out by experience, I must leave to the reader to decide.

The above discussion of tone intensities naturally leads us to take up the theoretical aspects of the fact frequently ob-

The disappearance of a higher tone

served by experimenters that in a combination of a lower and a higher tone the latter is sometimes entirely inaudible, provided, of course, that it is physically much weaker than the former. The reverse, however,

that is, the disappearance of a physically weak low tone when sounded together with a strong higher tone, has hardly been observed. The phenomenon in question can, perhaps, be most easily observed with such ratios at $1:2$, $2:3$, or $1:3$. Let us study, then, one of these ratios, say $1:2$, from the theoretical point of view.
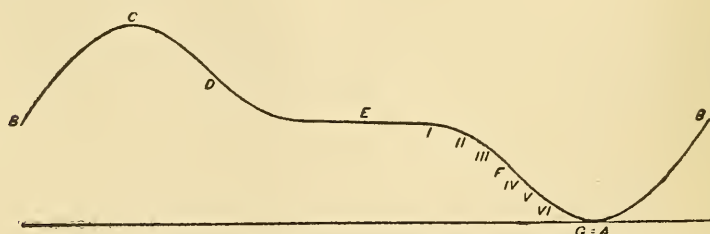
Fig. 26.   The combination 1 and 2, unequal amplitudes

Let us combine two sinusoids according to the following equation:

$$f(x) = 2\sin x + \sin 2x.$$

The combination 1 and 2, when 2 is comparatively weak — That is, the amplitude of the sinusoid of the shorter period is one-half of the amplitude of the sinusoid of the longer period. Figure 26 shows the curve representing the stirrup movement, and the accompanying table shows the exact numerical values of those points of the curve which, as we shall see, are of particular importance to us, that is, the maxima and minima, and the points of inflection. These values are easily found in this particular case. To find the maxima and minima, we have to set the first derivative of the above function equal to zero and solve the equation for x; for the maxima and minima are those points where the tangential angle or differential coefficient is zero.

$$f'(x) = 2\cos x + 2\cos 2x = 0.$$

To find the points of inflection, we have to set the second derivative equal to zero and solve the equation for x; for the points of inflection are those points of the curve where the tangential angle neither increases nor decreases.

$$f''(x) = -2\sin x - 4\sin 2x = 0.$$

The purely arithmetical work I do not care to perform here. The table shows its results. It is plain that, if we represent the successive positions of the partition according to the same rules as formerly employed, we find that only one tone can become audible, the tone 1. The tone 2 has disappeared because its addition does not increase the number of the maxima and minima of the compound curve (Fig. 26), but merely influences its shape. However interesting this insight may be into the fact that a weak higher tone added to a strong lower tone may be entirely inaudible, the present theoretic result is not quite satisfactory. It is somewhat un-

satisfactory because it seems improbable that the higher octave should become inaudible as soon as its amplitude is decreased to one-half of the amplitude of the lower tone. It seems, judging from experimental experience, that the higher octave must be weakened by far more, in order to become entirely inaudi-

INTERVAL 1 : 2, AMPLITUDES 2 : 1

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	0	0	2598	B	2598
Max.	+ 2598	600	5196	C	2598
Inf.	+ 1125	1045	3723	D	1473
Inf.	0	1800	2598	E	1125
Inf.	— 1125	2555	1473	F	1125
Min.	— 2598	3000	0	G = A	1473
Inf.	0	3600	2598	B	1473

ble. Now, to correct the above theoretic result, we cannot make use of the previous considerations concerning the influence of the tapering of the partition. As long as there is an initial section, however short, jerked down and up twice during the period, the result of tapering may be the lengthening of this section and a corresponding increase of the relative intensity of the higher tone. But when there is no initial section at all which moves twice, no tapering of the partition can create one. Let us, therefore, recall the other provisional assumptions.

The second of our provisional assumptions is that the partition is perfectly inelastic, that is, not offering any re-

The second provisional assumption recalled

sistance to a displacement until either of the limits is reached, and then offering absolute resistance. Now, does our anatomical knowledge warrant such an assumption? The most striking fact derived from an anatomical study of the organ is the absence of any solid body which might serve to interfere suddenly, abruptly, with a yielding movement of the partition in either direction. Even the analogy with the leather seat of a chair is hardly admissible if we mean thereby a flabby, wrinkled piece of leather. The analogy probably holds good only if we imagine the leather in such a condition as we find it in a new, unused chair, occupying a perfect plane, being practically free, however, from any stresses as long as no weight is resting upon it, yielding to a certain extent if a certain weight is placed upon it, but not yielding in proportion to the weight if the weight is increased. It is probably in a similar manner that the partition resists pressure. What determines the limit of yielding must be the partition's own elasticity. But let us always remember that there is no elastic force—no stress—in the partition while in its normal position, that its elastic force is the result of a displacement in either direction, that this elastic force increases much more rapidly than the displacement, and that therefore a constant increase of pressure on any point of the partition does not cause a constant movement of this point, but a movement first rapid, then quickly decreasing in velocity. Figure 27 is a graphic representation of such a function under the arbitrary assumption—which, perhaps, may be regarded as a rough approximation to the actual conditions—that the elastic force of the partition increases proportionally to the tangent of its displacement. The abscissæ represent the increasing pressure, the ordinates

the corresponding displacements of the partition. We notice, then, that there is a practical limit of yielding, that an increase of pressure beyond a certain point is practically ineffective, does not cause any further displacement to speak of.

There can be no doubt that the assumption of a relation existing between the displacement of the partition and the pressure, similar to the relation between an angle and its tangent—however rough the approximation to the facts—is
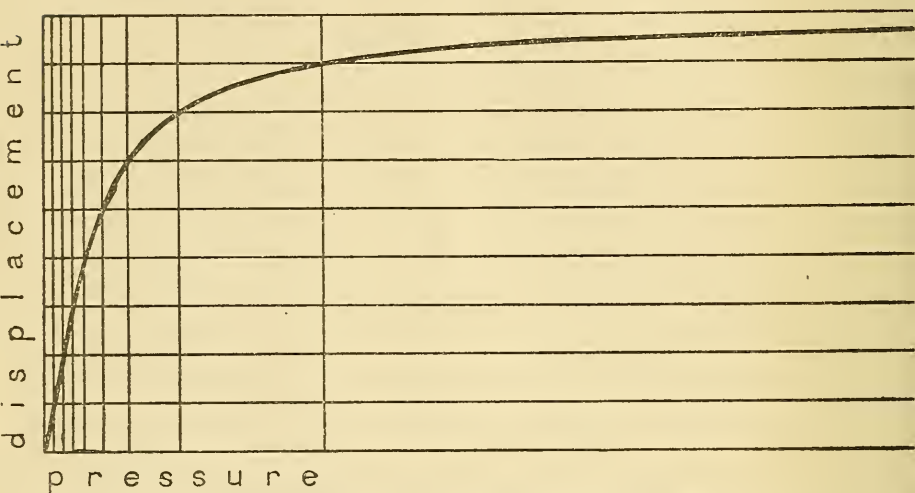

Fig. 27.   The probable relation between pressure and displacement of the partition

much better adapted to the anatomical facts than the second provisional assumption. Of course, the second provisional assumption simplifies greatly the graphic representation of the successive positions of the partition, but at the cost of all accuracy. Wherever the approximation thus possible is sufficient for our purposes, we shall, of course, continue to work under that simpler assumption. But let us now apply the latter assumption to our problem of representing the successive positions of the partition which correspond to the stirrup

movement of the curve in figure 26. Let us disregard, however, the varying width of the partition, in order to avoid too much complication. We shall again assume the partition to be of uniform width, without, however, forgetting the fact that this is an arbitrary simplification of the conditions.

Imagine that the whole partition is in its normal position, free of any stress, and that the stirrup begins an outward movement of the form of the curve from E to G in figure 26. We see from the curve **The significance** that the stirrup moves at first very slowly, **of a point of** then gradually more and more quickly un- **inflection** til at F, the point of inflection, it moves with the greatest velocity. Now, a simple consideration will make it plain to us that the pressure acting upon the initial part of the partition must be dependent on, probably be proportional to the velocity of the stirrup. If the velocity of the stirrup movement were extremely small, no point of the partition would move more readily than any other, and consequently none of them would move to a considerable extent; but the fluid would every time and all the time flow through the opening at the end of the tube which we called the safety valve, because there would then be practically no friction at any point within the tube, and an infinitesimal elastic force of displacement could keep the partition in place. On the other hand, if the velocity of the stirrup movement is not very small, the points of the partition near the windows receive the greatest push from the fluid, farther points only a slighter push, very quickly diminishing with increasing distance, and at some distance away the push could be regarded as practically infinitesimal; all this as the result of the friction of the fluid in the narrow tube, the total influence of which is the greater the longer the column of fluid in question, measuring this column from the windows.

As the stirrup moves away from E, the initial part of the
partition yields upwards, as shown in figure 28 at I. By I,
II, and so forth, are meant successive moments between E
and G in figure 26. The increasing velocity of the stirrup re-
sults at II in an increased pressure at all the points of the
partition which had yielded at I. Therefore, at II in figure 28
these points are somewhat farther displaced than they were at
I, but not proportional to the increase of the velocity of the
stirrup but much less, according to figure 27. At the same
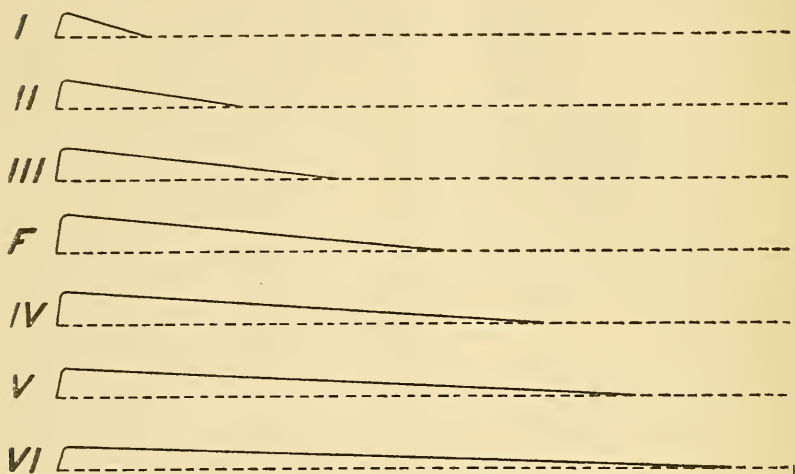
Fig. 28.   Seven successive positions of the partition, three preceding
and three following an inflection point (F)

time we notice that the part of the partition which has now
yielded extends much farther to the right at II than at I;
for the stirrup has displaced much more fluid at II than at the
earlier moment I, and the slight increase in the displacement
of those parts of the partition which were already displaced
at I, can not nearly make room for all this fluid. Therefore the

spreading of the displacement lengthwise over the partition.
At III the velocity of the stirrup is still greater than at II.
Therefore we notice again a slight increase in the displace-
ment of the initial part of the partition. But as the stirrup
approaches F, this increase of displacement of the initial parts
must become less; for the velocity of the stirrup is now nearly
constant, its increase very slight, and the increase of displace-
ment is in any case much less than proportional to the increase
of velocity, according to figure 27. As soon as the stirrup passes
F, its velocity begins to decrease. Immediately the press-
ure on the whole piece of the partition which has yielded de-
creases; and this whole piece, therefore begins to move
slowly back by its elasticity in the direction of its normal
position. It is clear, however, from figure 27 that even a
considerable decrease of the velocity of the stirrup causes only
a slight decrease of the displacement until the stirrup ap-
proaches G, when its velocity approaches zero and the part
of the partition in question can move more rapidly by its
elasticity since it has no longer to overcome much pressure
caused by the stirrup. It does not follow, however, that any
point of the partition has returned to its normal position by
the time the stirrup reaches G. The initial sections have
merely moved in the direction of their normal position. And
meanwhile, new points of the partition to the right must
have yielded upwards to make room for the fluid being dis-
placed all the time by the stirrup in moving towards G.
Three positions of the partition between F and G are shown
in figure 28 at IV, V, and VI.

One of the consequences of the decrease of pressure on the partition at the point of inflection between a maximum and a

preceding or following minimum of the curve consists in the fact that the partition does not move up and down so suddenly as it appeared from our previous graphic representations. We had to point out this fact before in mentioning the irregularity with which stimuli often seem to be received by the nerve ends according to our simplified graphic representation. The exact time when a stimulus—a shock, as we called it—is received we now find to be dependent also on the location of each inflection point, not merely on the temporal location of the maxima and minima. Unfortunately, however, we can not determine the time of each shock with certainty even now, taking into account the inflection point. This important question of theoretical detail must be left open for future investigation.

Another consequence of the decrease of pressure on the partition marked by any point of inflection consists in the fact that a double movement—up and down—of the partition may result, not only from an alternation of maxima and minima of a curve, but also from an alternation of inflection points marking an increasing and decreasing velocity of the stirrup. This means that the number of shocks received by the nerve ends during one period of the curve may exceed the total number of maxima (or minima) in case any part of the curve from a maximum to a minimum or from a minimum to a maximum contains more than a single point of inflection. An example will be given at once.

Let us return to the theoretical analysis of the whole curve in figure 26. From A to C the stirrup moves inwards, pushing down a certain length of the partition. The initial part of this length, however, begins a slow upward movement as soon as the velocity of the stirrup begins to decrease, at B. The same part moves up more quickly when, at C, the stirrup reverses its movement and begins to pull

The successive positions of the partition corresponding to figure 26

it upward. We therefore see at B in figure 29 the initial two sections in an extreme downward position. At C, we see them
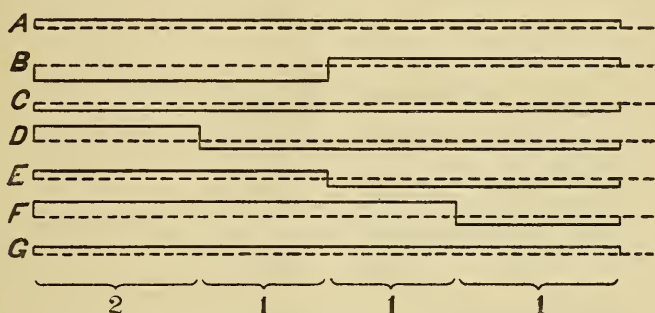

Fig. 29. The combination 1 and 2. Compare figure 26

only in a medium downward position, and at the same time we find the following two sections of the partition in a similar downward position since the stirrup has continued, from B to C, to move inwards. It is plain that to take into account, in our graphic representation, only two kinds of displacements in either direction, an extreme and a medium one, is again an artificial simplification, introduced merely to suit our momentary needs, in spite of the fact that thus we lose sight of some of the details of the movement. Actually, the movement probably occurs rather in the form of figure 28. But the simplification used in figure 29 not only renders the drawing of the figure

easier, but also contributes towards a readier comprehension
of the significance of the graphic representation, towards a
quicker reading off of the tones to be heard.

At D we see the first section in an extreme upward position
since the stirrup has moved outwards and has reached a max-
imum velocity. At E, the first section has returned to a me-
dium displacement since the velocity of the stirrup has reached
a minimum. At the same time the second section of the par-
tition has moved upwards as a result of the continued outward
movement of the stirrup. At F we find the initial three sec-
tions of the partition in an extreme upward position; for the
stirrup has continued to move outwards and has also reached
a maximum of velocity. At G all four initial sections of the
partition are in an upward position since the stirrup has con-
tinued to move outwards. But they are only in a medium
displacement since the velocity of the stirrup has again reach-
ed a minimum.

Looking now over the four columns in figure 29, we notice
that the first shows an extreme upward position of this section
of the partition at F, a medium upward
position at G=A, an extreme downward
position at B, a medium downward posi-
tion at C, an extreme upward position at
D, a medium upward position at E, an ex-
treme upward position again at F. This section of the parti-
tion, therefore, has moved up and down twice during the pe-
riod, the second upward movement occurring between E and
F. It is quite probable, then, that the nerve ends located on
this section receive two shocks during the period. The second
section of the partition has an extreme upward position at F,
a medium upward position at G=A, an extreme downward
position at B, a medium downward position at C and D,
and a medium upward position at E. It follows that
this section moves up and down only once during the pe-

**De we hear
both tones
2 and 1?**

riod, and that the nerve ends located there receive only one shock during the period. The third section has an extreme upward position at F, a medium upward position at G=A and also at B, a medium downward position at C, D, and E. The nerve ends of this section receive therefore one shock during the period. The fourth section has a medium upward position at G=A and at B, a medium downward position at C, D, E, and F. The nerve ends of this section receive therefore one shock during the period. It is plain, then, that from our theory we must expect to hear the tone 2 as well as the tone 1, the former conveyed by the first, the latter by the three following sections of the partition.

To determine the relative intensities of the tones heard, we have to compare the length of the initial section of the partition with the total length of the three **Sixth provision-** following sections when added together. **al assumption** For simplicity's sake, let us make this comparison again under the third and fourth provisional assumptions, and also under a new assumption, namely, that the fluid for which room is made or whose room is taken by a move of the partition from a medium to an extreme (or the reverse) displacement on the same side (either above or below the normal position) is a negligible quantity. That this assumption simplifies our representation of the successive positions of the several sections of the partition is clear, since we may thus take the length of each section proportional to the ordinate difference of the corresponding points of the curve. For instance, the third and fourth sections in figure 29, which move down at C, would be longer than proportional to the ordinate difference of the points B and C in figure 26 if the fluid displaced by the first and second sections in moving from an extreme position at B to a medium displacement at C were not a negligible quantity. In the latter case, the fluid displaced by the first and second sections during the

time from B to C would have to be made room for by the third
and fourth sections, which, then, by necessity would extend
farther to the right than in proportion to the stirrup movement
from B to C.   To take this into account would extraordinarily
complicate the graphic representation without offering, at
present, a correspondingly great advantage.   This additional
extension of the third and fourth sections to the right could
be but slight since the amount of fluid in question would be
but slight.   This becomes clear from a glance at figure 27.
We have learnt from this figure that some pressure added to
a given pressure does not cause a proportional, but a much
smaller increment to be added to the previous displacement of
the partition; and thus the amount of fluid in question may be
entirely neglected without depriving us of the right to regard
our representation as an approximation to the actual positions
of the partition sections.

We may, then, under the third, fourth, and sixth pro-
visional assumptions, regard the relative intensities of the
tones as proportional to the ordinate dif-
ferences in the table belonging to figure
The relative 26.   We find in the table the value 1473 as
intensities of expressing the ordinate difference of C and
2 and 1 D, the value 1125 of D and E, 1125
of E and F, and 1473 of F and G, the sum of these last three being
3723.   Therefore, under the above simplifying assumptions,
the relative intensity of the tone 2 compared with 1 is about
as fifteen to thirty-seven.

Let us now apply our theory to the ratio of the vibration
rates 5 : 8.   The curve in figure 30 represents the function

$$f(x) = \sin 5x + \sin 8x.$$

The table below contains all the abscissa and ordinate
values of the maxima and minima as well
The combina- as of the inflection points of the curve.
tion 5 and 8. The inflection points are computed as the
Equal ampli- maxima and minima of the first derivative
tudes of stirrup curve, represented by the function
movement

$$f'(x) = 5\cos 5x + 8\cos 8x.$$

It is impossible, in this case, to apply the simple method of

Fig. 34

Fig. 32

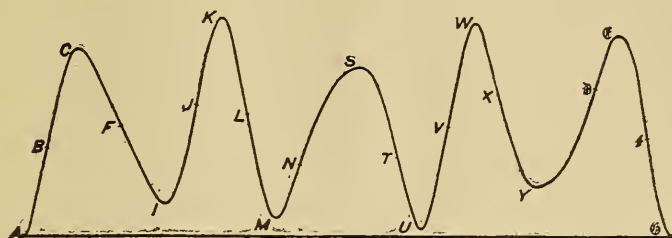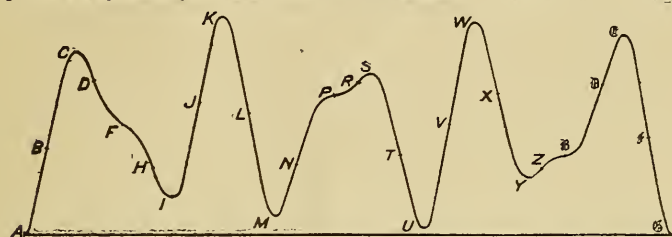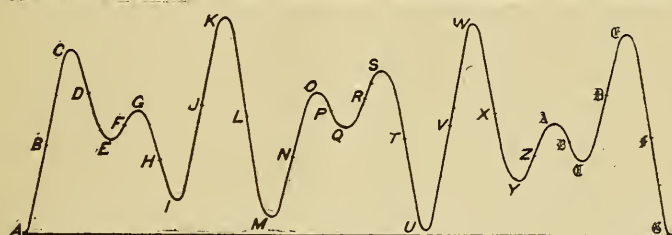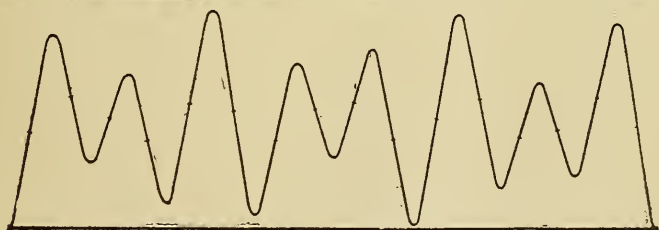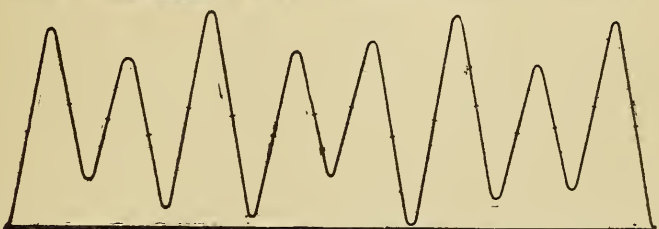Fig. 30

Fig. 36

Fig. 38

The combination 5 and 8 with different amplitude ratios

finding the corresponding ordinate and abscissa values of the maxima and minima of these two functions by making their derivatives equal to zero and solving the resultant equations

INTERVAL 5:8, EQUAL AMPLITUDES

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	0	0	199	V	188
Max.	+ 188	131	387	W	188
Inf.	+ 24	249	223	X	164
Min.	− 100	385	99	Y	124
Inf.	− 51	474	148	Z	49
Max.	+ 3	576	202	𝔄	54
Inf.	− 29	661	170	𝔅	32
Min.	− 61	740	138	ℭ	32
Inf.	+ 59	872	258	𝔇	120
Max.	+ 167	983	366	𝔈	108
Inf.	− 18	1116	181	𝔉	185
Min.	− 199	1244	0	A	181
Inf.	− 36	1367	163	B	163
Max.	+ 137	1504	336	C	173
Inf.	+ 61	1603	260	D	76
Min.	− 26	1725	173	E	87
Inf.	0	1800	199	F	26
Max.	+ 26	1875	225	G	26
Inf.	− 61	1997	138	H	87
Min.	− 137	2096	62	I	76
Inf.	+ 36	2233	235	J	173
Max.	+ 199	2356	398	K	163
Inf.	+ 18	2484	217	L	181
Min.	− 167	2617	32	M	185
Inf.	− 59	2728	140	N	108
Max.	+ 61	2860	260	O	120
Inf	+ 29	2939	228	P	32
Min.	− 3	3024	196	Q	32
Inf.	+ 51	3126	250	R	54
Max.	+ 100	3215	299	S	49
Inf.	− 24	3351	175	T	124
Min.	− 188	3469	11	U	164
Inf.	0	3600	199	V	188

for x. This is impossible because the equations to be solved would be of the eighth degree. We have to use, therefore, the only method left, however great our sacrifice of time, and to calculate directly a sufficiently large number of values from which we then select the largest and smallest. In this way the values of the table have been obtained. By adding 199 to each of the values of the first column we get the third column, which offers the advantage of containing only positive ordinates. This procedure is equivalent to selecting a different horizontal coordinate, which is always dependent on our choice. The ordinate value zero, thus obtained, is the one which belongs to point A in figure 30. The successive positions of the partition corresponding, under the sixth provisional assumption, to all the maxima, minima, and inflection points of the curve are shown in figure 31.

Let us at once examine the movements of the three sections, the fiftieth, the fifty-first, and the fifty-second.* At A, we find these sections occupying a medium upward position. From A to B they move down. From B to C they begin to move up. From C to D they continue to move up. From D to E they begin to move down

What tones do we hear? The tone 8

and continue to move down until G. From G to H they move up, completing thus the second down and up movement. From H to J they move down, and from J to L up, completing the third down and up movement. From L to N they move down, and from N to Q up, completing the fourth down and up movement. From Q to R they move down, and from R to T up, completing the fifth down and up movement. From T to V they move down, and from V to X up, completing the sixth

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* For a perfect understanding of the details, the reader will have to draw figure 31 (and the similar figures following) for himself on a larger scale, and to inscribe the exact values as derived from each corresponding table.
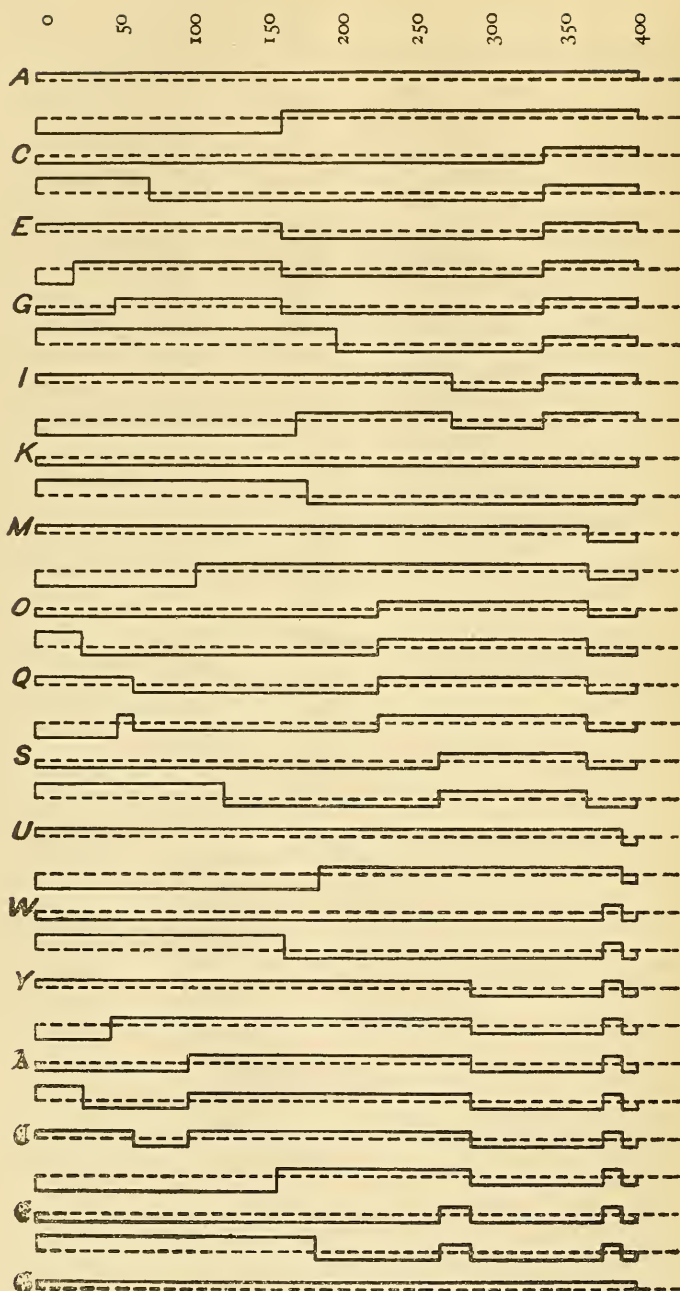
Fig. 31.   Compare figure 30

down and up movement. From X to 𝔄 they move down, and from 𝔄 to ℭ up, completing the seventh down and up movement. From ℭ to 𝔇 they move down, and from 𝔇 to 𝔉 up, completing the eighth down and up movement. From 𝔉 to 𝔊=A they begin to move down and continue to move down after A, as we have seen.

The movements of the forty-nine initial sections are so similar to those of the three sections just discussed that we convince ourselves easily that the nerve ends located there receive the same number of shocks during the period.

The fifty-third and fifty-fourth sections move down from 𝔉 to B, and up from B to D. Down from D to G, and up from G to H. Down from H to J, and up from J to L. Down from L to N, and up from N to Q. Down from Q to R, and up from R to T. Down from T to V, and up from V to X. Down from X to 𝔄, and up from 𝔄 to ℭ. Down from ℭ to 𝔇, and up from 𝔇 to 𝔉. The nerve ends located on these sections therefore receive eight shocks during the period.

The ten sections from the fifty-fifth to the sixty-fourth move down from 𝔉 to B, and up from B to D. Down from D to G, and up from G to H. Down from H to J, and up from J to L. Down from L to N, and up from N to Q. Down from Q to S, and up from S to T. Down from T to V, and up from V to X. Down from X to 𝔄, and up from 𝔄 to ℭ. Down from ℭ to 𝔇, and up from 𝔇 to 𝔉. The nerve ends located on these sections therefore receive eight shocks during the period.

The twelve sections from the sixty-fifth to the seventy-sixth move down from 𝔉 to B, and up from B to D. Down from D to G, and up from G to H. Down from H to J, and up from J to L. Down from L to N, and up from N to T. Down from T to V, and up from V to X. Down from X to 𝔇, and up from 𝔇 to 𝔉. The nerve ends located on these sections therefore receive six shocks during the period.

The tone 6

The twenty seven sections from the seventy-seventh to the

hundred and third move down from $\mathfrak{F}$ to C, and up from C to

H. Down from H to J, and up from J to

The tone 5   L. Down from L to N, and up from N to

T. Down from T to V, and up from V to

X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive five shocks during the period.

The five sections from the hundred and fourth to the hundred and eighth move down from $\mathfrak{F}$ to C, and up from C to H. Down from H to J, and up from J to L. Down from L to N, and up from N to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive five shocks during the period.

All the following sections to the two hundred and sixty-seventh move down and up five times during the period. Let us study in detail only the movements of the last few of this group. The sections from the two hundred and twenty-eighth to the two hundred and sixty-seventh move down from A to C, and up from C to I. Down from I to K, and up from K to M. Down from M to S, and up from S to U. Down from U to W, and up from W to Y. Down from Y to $\mathfrak{E}$, and up from $\mathfrak{E}$ to $\mathfrak{G}=A$. The nerve ends located on these sections therefore receive five shocks during the period.

The seven sections from the two hundred and sixty-eighth to the two hundred and seventy-fourth move down from Y

to C, and up from C to I. Down from I

The tone 3   to K, and up from K to M. Down from

M to W, and up from W to Y. The nerve-ends located on these sections therefore receive three shocks during the period.

The fourteen sections from the two hundred and seventy-fifth to the two hundred and eighty-eighth move down

from Y to C, and up from C to M. Down

The tone 2   from M to W, and up from W to Y. The

sections from the two hundred and eighty-ninth to the three hundred and thirty-sixth move down from

A to C, and up from C to M. Down from M to W, and up from W to ℭ=A. The sections from the three hundred and thirty-seventh to the three hundred and sixty-sixth move down from A to K, and up from K to M. Down from M to W, and up from W to ℭ=A. The sections from the three hundred and sixty-seventh to the three hundred and seventy-sixth move down from A to K, and up from K to U. Down from U to W, and up from W to ℭ=A. All these sections therefore receive two shocks during the period.

The sections from the three hundred and seventy-seventh to the three hundred and eighty-seventh move down from U **The tone 1** to K, and up from K to U. The sections from the three hundred and eighty-eighth to the three hundred and ninety-eighth move down from A to K, and up from K to ℭ=A. All these sections therefore receive one shock during the period.

The relative intensities of the several tones, if we accept the third, fourth, and sixth provisional assumptions for this case, are shown in the following table, **The relative intensities** which contains the number of partition sections conveying each tone in absolute numbers as well as in percentages.

Tones	8	6	5	3	2	I
Intensities	64	12	191	7	102	22
Percent-ages	16.1	3.0	48.0	1.8	25.6	5.5

Let us now apply our theory to the same ratio of the vibration rates, but with different amplitudes of the two sinusoids. The curve in figure 32 represents the function

The combination 5 and 8. The amplitude of 8 is decreased

$$f(x) = 2\sin5x + \sin8x.$$

This signifies that the stirrup movement eight has an amplitude which is only one-half of the amplitude of the stirrup movement five. The table below contains all the abscissa and

ordinate values of the maxima and minima and of the inflection points of the curve.

INTERVAL 5:8, AMPLITUDES 2:1

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	o	o	298	V	281
Max.	+ 281	142	579	W	281
Inf.	+ 87	268	385	X	194
Min.	− 143	436	155	Y	230
Inf.	− 118	512	180	Z	25
Max.				𝔄	
Inf.	− 82	636	216	𝔅	36
Min.				ℭ	
Inf.	+ 110	846	408	𝔇	192
Max.	+ 248	962	546	𝔈	138
Inf.	− 34	1111	264	𝔉	282
Min.	− 298	1247	o	A	264
Inf.	− 62	1379	236	B	236
Max.	+ 200	1535	498	C	262
Inf.	+ 120	1638	418	D	80
Min.				E	
Inf.	o	1800	298	F	120
Max.				G	
Inf.	− 120	1962	178	H	120
Min.	− 200	2065	98	I	80
Inf.	+ 62	2221	360	J	262
Max.	+ 298	2353	596	K	236
Inf.	+ 34	2489	332	L	264
Min.	− 248	2638	50	M	282
Inf.	− 110	2754	188	N	138
Max.				O	
Inf.	+ 82	2964	380	P	192
Min				Q	
Inf.	+ 118	3088	416	R	36
Max.	+ 143	3164	441	S	25
Inf.	− 87	3332	211	T	230
Min.	− 281	3458	17	U	194
Inf.	o	3600	298	V	281

These values have been computed in the same manner as in the case immediately preceding. The successive positions of the partition corresponding, under the sixth provisional assumption, to the maxima, minima, and inflection points of the curve are shown in figure 33.
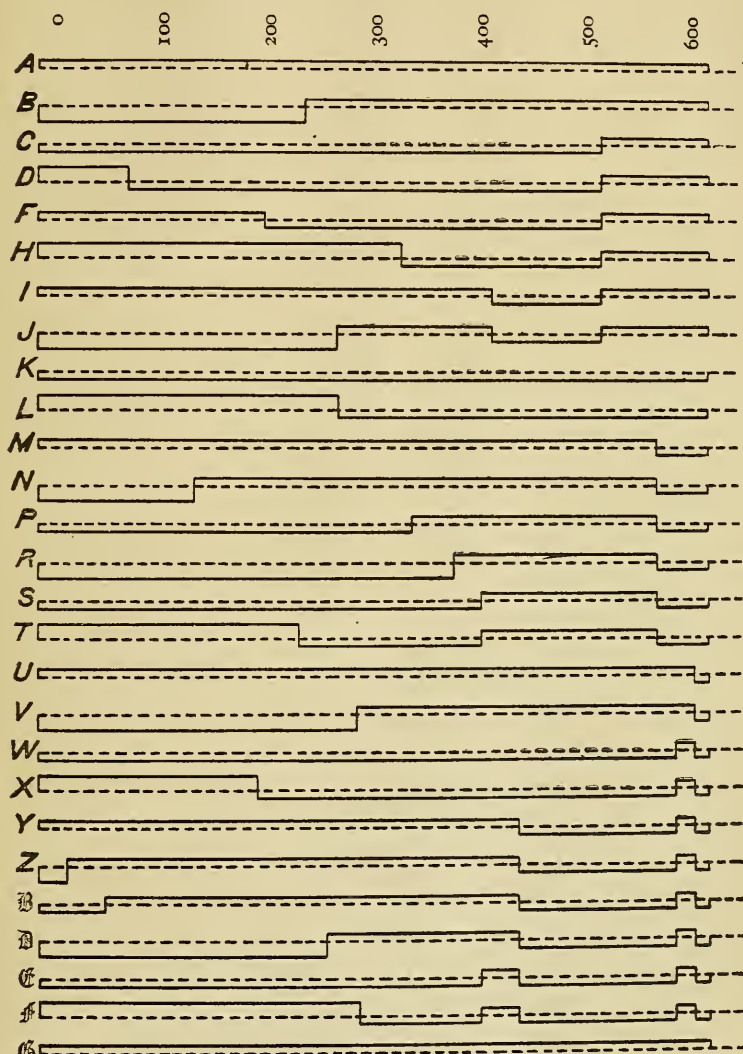

Fig. 33. The combination 5 and 8. Compare figure 32

Let us examine the movements of the twenty-five initial sections. From $\mathfrak{F}$ to B they move down, and from B to D up.

The tone 8
From D to F down, that is, from an extreme upward position to a medium upward position; and from F to H they move up again, that is, from a medium upward position to an extreme upward position. From H to J they move down, and from J to L up, completing thus the third down and up movement. From L to N they move down and from N to P up, completing thus the fourth down and up movement. From P to R they move down, and from R to T up, completing thus the fifth down and up movement. From T to V down, and from V to X up, completing thus the sixth down and up movement. From X to Z down, and from Z to $\mathfrak{B}$ up, completing thus the seventh down and up movement. From $\mathfrak{B}$ to $\mathfrak{D}$ down and from $\mathfrak{D}$ to $\mathfrak{F}$ up again. The nerve ends located on these twenty-five sections therefore receive eight shocks during the period, and accordingly, convey the sensation of the tone 8.

The thirty-six sections from the twenty-sixth to the sixty-first move down from $\mathfrak{F}$ to B, and up from B to D. Down from D to F, and up from F to H. Down from H to J, and up from J to L. Down from L to N, and up from N to P. Down from P to R, and up from R to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive seven shocks during the period. But, in accordance with previous considerations, it is highly improbable that they could convey the sensation of the tone 7. When seven shocks are received in time intervals identical with those of the tone 8, and when the eighth shock, at the moment Z, chances to be omitted, it is rather to be expected that the tone 8 is heard, only with a little pause or, perhaps, merely a diminution of intensity at the moment Z. The sensation conveyed

by these nerve ends, then, is probably the tone 8 slightly beating, that is, being characterized by a slight roughness.

The nineteen sections from the sixty-second to the eightieth move down from $\mathfrak{F}$ to B, and up from B to D. Down from D to F, and up from F to H. Down from H to J, and up from J to L. Down from L to N, and up from N to P. Down from P to R, and up from R to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive seven shocks during the period; but, here as above, it is highly improbable that they could convey, merely because of the omission of the stimulus at Z, the sensation of the tone 7 instead of 8. Most probably the tone heard is 8 with a slight roughness.

The fifty-eight sections from the eighty-first to the one hundred and thirty-eighth move down from $\mathfrak{F}$ to B, and up from B to H. Down from H to J, and up **The tone 6** from J to L. Down from L to N, and up from N to P. Down from P to R, and up from R to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive six shocks during the period.

The fifty-six sections from the one hundred and thirty-ninth to the one hundred and ninety-fourth move down from $\mathfrak{F}$ to B, and up from B to H. Down from **The tone 5** H to J, and up from J to L. Down from L to R, and up from R to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The nerve ends located on these sections therefore receive five shocks during the period.

All the following sections to the three hundred and ninety-first move down and up five times during the period. Let us examine only the last twenty-five of this group. They move

down from A to C and up from C to I. Down from I to K, and up from K to M. Down from M to S, and up from S to U. Down from U to W, and up from W to Y. Down from Y to ℭ, and up from ℭ to ℭ=A.

The nine sections from the three hundred and ninety-second to the four hundredth move down from Y to C, and **The tone 3** up from C to I. Down from I to K, and up from K to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive three shocks during the period.

The sections from the four hundred and first to the four hundred and twenty-fourth move down from Y to C, and up **The tone 2** from C to M. Down from M to W, and up from W to Y. The sections from the four hundred and twenty-fifth to the four hundred and ninety-eighth move down from A to C, and up from C to M. Down from M to W, and up from W to ℭ=A. The sections from the four hundred and ninety-ninth to the five hundred and forty-sixth move down from A to K, and up from K to M. Down from M to W, and up from W to ℭ=A. The sections from the five hundred and forty-seventh to the five hundred and sixty-second move down from A to K, and up from K to U. Down from U to W, and up from W to ℭ=A. The nerve ends located on these sections of the partition therefore receive two shocks during the period.

The sections from the five hundred and sixty-third to the five hundred and seventy-ninth move down from U to K, and **The tone 1** up from K to U. The sections from the five hundred and eightieth to the five hundred and ninety-sixth move down from A to K, and up from K to ℭ=A. The nerve ends located on these sections therefore receive one shock during the period.

The relative intensities of the several tones, if we accept

the third, fourth, and sixth provisional assumptions, are shown

The relative intensities

in the following table, which contains the number of partition sections conveying each tone in absolute numbers as well as in percentages.

Tones	8, smooth	8, rough	6	5	3	2	1
Intensities	25	55	58	253	9	162	34
Percent-ages	4.2	9.2	9.7	42.5	1.5	27.2	5.7

Since in the case just studied the higher of the two primary tones, though weak, is yet audible, let us still further change the relative intensities of the objective tones in favor of the lower one. The curve in figure 34 represents the function

$$f(x) = 3\sin 5x + \sin 8x.$$

The combination 5 and 8. Amplitude of 8 still less

This signifies that the stirrup movement eight has an amplitude which is only one-third of the amplitude of the stirrup movement five. The table below contains all the abscissa and ordinate values of the maxima and minima and of the inflection points of the curve.

INTERVAL 5:8, AMPLITUDES 3:1

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	o	o	397	V	376
Max.	+ 376	149	773	W	376
Inf.	+ 114	283	511	X	262
Min.	− 219	483	178	Y	333
Inf.				Z	
Max.				𝔄	
Inf.				𝔅	
Min.				ℭ	
Inf.	+ 140	820	537	𝔇	359
Max.	+ 336	949	733	𝔈	196
Inf.	− 41	1106	356	𝔉	377
Min.	− 397	1250	o	A	356
Inf.	− 77	1389	320	B	320
Max.	+ 281	1558	678	C	358
Inf.				D	
Min.				E	
Inf.	o	1800	397	F	281
Max.				G	
Inf.				H	
Min.	− 281	2042	116	I	281
Inf.	+ 77	2211	474	J	358
Max.	+ 397	2350	794	K	320
Inf.	+ 41	2494	438	L	356
Min.	− 336	2651	61	M	377
Inf.	− 140	2780	257	N	196
Max.				O	
Inf.				P	
Min.				Q	
Inf.				R	
Max.	+ 219	3117	616	S	359
Inf.	− 114	3317	283	T	333
Min.	− 376	3451	21	U	262
Inf.	o	3600	397	V	376

The successive positions of the partition corresponding, under the sixth provisional assumption, to the maxima, minima, and inflection points of the curve are shown in figure 35.
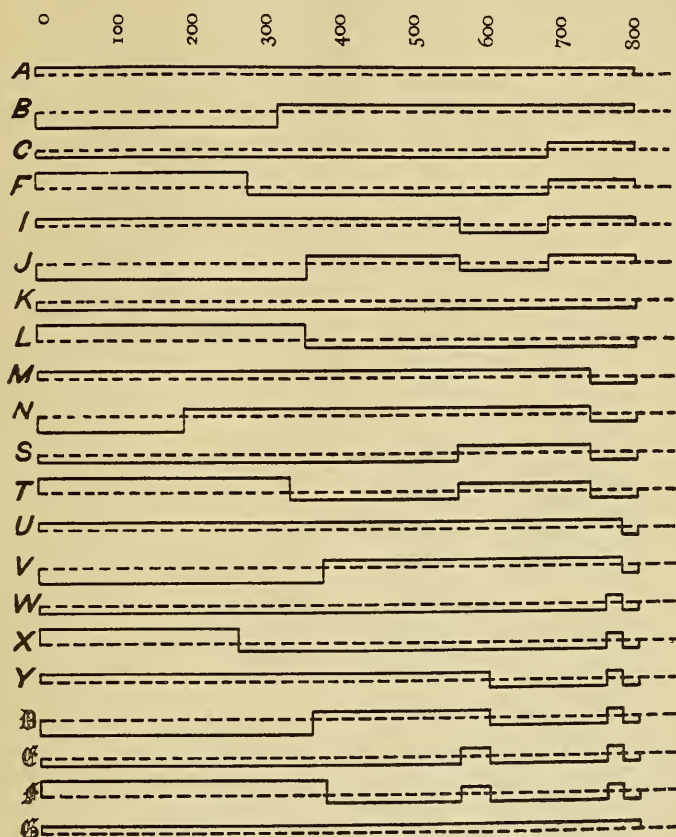

Fig. 35.   The combination 5 and 8.   Compare figure 34

The five hundred and fifty-five initial sections of the partition move down and up five times during the period. Let us here closely examine only the one hundred and ninety-six initial sections and the one hundred and seventy-eight most distant sections of this group. The initial sections move

The tone 5

down from $\mathfrak{F}$ to B, and up from B to F. Down from F to J, and up from J to L. Down from L to N, and up from N to T. Down from T to V, and up from V to X. Down from X to $\mathfrak{D}$, and up from $\mathfrak{D}$ to $\mathfrak{F}$. The sections from the three hundred and seventy-eighth to the five hundred and fifty-fifth move down from A to C, and up from C to I. Down from I to K, and up from K to M. Down from M to S, and up from S to U. Down from U to W, and up from W to Y. Down from Y to $\mathfrak{E}$ and up from $\mathfrak{E}$ to $\mathfrak{G}{=}A$. The nerve ends of all these sections therefore receive five shocks during the period.

The seven sections from the five hundred and fifty-sixth to the five hundred and sixty-second move down from Y to

The tone 3 C, and up from C to I. Down from I to K, and up from K to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive three shocks during the period.

The sections of the partition from the five hundred and sixty-third to the five hundred and seventy-second move down

The tone 2 and up twice during the period. Let us here examine only the sections from the five hundred and sixty-third to the five hundred and ninety-fifth. They move down from Y to C, and up from C to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive two shocks during the period.

The partition sections from the seven hundred and fifty-third to the seven hundred and seventy-third move down from

The tone 1 U to K, and up from K to U. The sections from the seven hundred and seventy-fourth to the seven hundred and ninety-fourth move down from A to K, and up from K to $\mathfrak{G}{=}A$.

All the nerve ends on these sections therefore receive one shock during the period.

The relative intensities

The relative intensities of the several tones under the third, fourth, and sixth provisional assumptions are shown in the following table.

Tones	5	3	2	1
Intensities	555	7	190	42
Percentages	69.9	.9	23.9	5.3

Having studied the effect of changing the relative intensities of the objective tones in favor of the lower one, we shall now investigate the effect of increasing the intensity of the higher objective tone. The curve in figure 36 represents the function

$$f(x) = \sin 5x + 2\sin 8x.$$

The combination 5 and 8. The amplitude of 8 is greater than of 5

The stirrup movement eight has an amplitude which is twice the amplitude of the stirrup movement five. The table below contains the abscissa and ordinate values of the maxima, minima, and inflection points.

INTERVAL 5:8, AMPLITUDES 1:2

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	0	0	298	V	286
Max.	+ 286	123	584	W	286
Inf.	+ 55	237	353	X	231
Min.	− 190	360	108	Y	245
Inf.	− 49	460	249	Z	141
Max.	+ 102	568	400	𝔄	151
Inf.	− 26	669	272	𝔅	128
Min.	− 152	767	146	ℭ	126
Inf.	+ 60	886	358	𝔇	212
Max.	+ 262	996	560	𝔈	202
Inf.	− 20	1120	278	𝔉	282
Min.	− 298	1241	0	A	278
Inf.	− 40	1359	258	B	258
Max.	+ 229	1484	527	C	269
Inf.	+ 60	1588	358	D	169
Min.	− 120	1702	178	E	180
Inf.	0	1800	298	F	120
Max.	+ 120	1898	418	G	120
Inf.	− 60	2012	238	H	180
Min.	− 229	2116	69	I	169
Inf.	+ 40	2241	338	J	269
Max.	+ 298	2359	596	K	258
Inf.	+ 20	2480	318	L	278
Min.	− 262	2604	36	M	282
Inf.	− 60	2714	238	N	202
Max.	+ 152	2833	450	O	212
Inf.	+ 26	2931	324	P	126
Min.	− 102	3032	196	Q	128
Inf.	+ 49	3140	347	R	151
Max.	+ 190	3240	488	S	141
Inf.	− 55	3363	243	T	245
Min.	− 286	3477	12	U	231
Inf.	0	3600	298	V	286

The successive positions of the partition corresponding, under the sixth provisional assumption, to the maxima, minima, and inflection points of the curve are shown in figure 37.

The two hundred and forty initial sections move down and up 8 times during the period. Let us here examine only **The tone 8** the nine most distant sections of this group, from the two hundred and thirty-second to the two hundred and fortieth. They move down from A to B, and up from B to E. Down from E to G, and up from G to I. Down from I to J, and up from J to L. Down from L to O, and up from O to Q. Down from Q to S, and up from S to T. Down from T to V, and up from V to Y. Down from Y to 𝔄, and up from 𝔄 to ℭ. Down from ℭ to 𝔈, and up from 𝔈 to 𝔊=A. The nerve ends located on these sections therefore receive eight shocks during the period.

The fourteen sections from the two hundred and forty-first to the two hundred and fifty-fourth do not move down from E to G. The nerve ends located on these sections do not, therefore, receive a shock between E and I, but receive the other seven shocks in the same manner as the two hundred and forty initial sections. For the same reasons as in the similar cases with which we have met before, it is not probable that these nerve ends convey the tone 7, but rather the tone 8 with a slight beat occurring once during the period, producing a slightly rough tone 8.

The sections of the partition from the two hundred and fifty-fifth to the four hundred and fifty-second move down and **The tone 5** up five times during the period. Let us examine those from the two hundred and fifty-fifth to the two hundred and fifty-eighth. They move down from A to B, and up from B to E. Down from E to J, and up from J to L. Down from L to O, and up from O to U. Down from U to V, and up from V
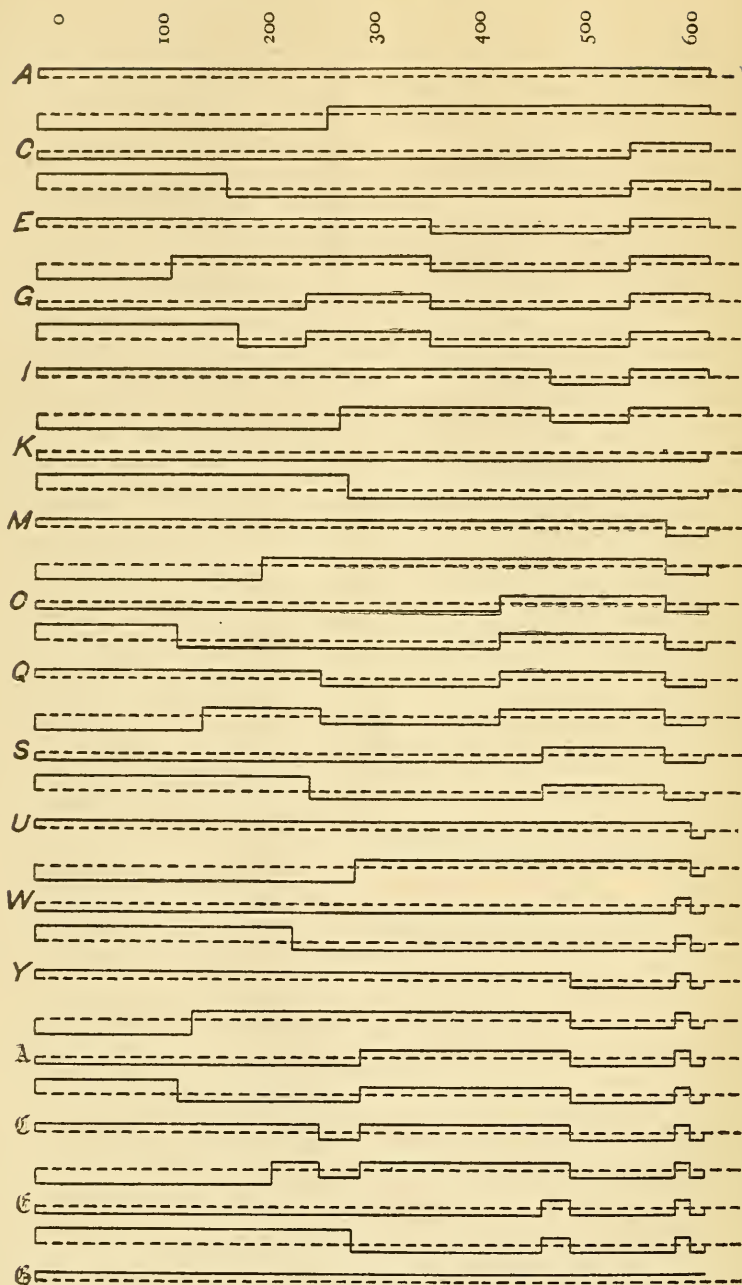
Fig. 37. The combination 5 and 8. Compare figure 36

to Y. Down from Y to 𝔄, and up from 𝔄 to 𝔊=A. The nerve ends located on these sections therefore receive five shocks during the period.

The sections from the four hundred and fifty-third to the four hundred and fifty-eighth move down from Y to C,

The tone 3   and up from C to I. Down from I to K, and up from K to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive three shocks during the period.

The sections of the partition from the four hundred and fifty-ninth to the five hundred and seventy-second move down

The tone 2   and up twice during the period. Let us examine, for example, the four hundred and fifty-ninth and the four hundred and sixtieth. They move down from Y to C, and up from C to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive two shocks during the period.

The sections of the partition from the five hundred and seventy-third to the five hundred and eighty-fourth move down

The tone 1   from U to K, and up from K to U. The sections from the five hundred and eighty-fifth to the five hundred and ninety-sixth move down from A to K, and up from K to A. The nerve ends located on these sections therefore receive one shock during the period.

The relative intensities   The relative intensities of the several tones under the third, fourth, and sixth provisional assumptions are shown in the following table.

Tones	8 smooth	8 rough	5	3	2	1
Intensities	240	14	198	6	114	24
Percent-ages. ..	40.3	2.4	33.2	1.0	19.1	4.0

The curve in figure 38 represents the function

$$f(x) = \sin 5x + 3\sin 8x.$$

The combination 5 and 8. The amplitude of 8 is three times that of 5

The stirrup movement eight has an amplitude three times as great as that of five. The table below contains the abscissa and ordinate values of the maxima, minima, and inflection points.

The successive positions of the partition corresponding, under the sixth provisional assumption, to the maxima, minima, and inflection points of the curve are shown in figure 39.

The four hundred and thirty-eight initial sections of the partition move down and up eight times during the period.

The tone 8

Let us examine those from the three hundred and eighty-sixth to the four hundred and thirty-eighth. They move down from A to C and up from C to E. Down from E to G, and up from G to I. Down from I to K and up from K to M. Down from M to O, and up from O to Q. Down from Q to S, and up from S to U. Down from U to W, and up from W to Y. Down from Y to 𝔄, and up from 𝔄 to ℭ. Down from ℭ to 𝔈, and up from 𝔈 to 𝔊=A. The nerve ends located on these sections therefore receive eight shocks during the period.

The sections from the four hundred and thirty-ninth to the four hundred and fifty-first move down and up only seven times, since they do not make a double movement between E and I. In accordance with our former considerations, however, in similar cases, it does not seem probable that the nerve ends located on these sections should convey any other tone than the tone 8 of a slight roughness.

The sections of the partition from the four hundred and fifty-second to the six hundred and forty-seventh move down

The tone 5

five times during the period. Let us examine those from the four hundred and fifty-second to the four hundred and eighty-ninth. They move down from A to C, and up from

INTERVAL 5:8, AMPLITUDES 1:3

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	o	o	398	V	385
Max.	+ 385	120	783	W	385
Inf.	+ 56	233	454	X	329
Min.	− 287	353	111	Y	343
Inf.	− 46	457	352	Z	241
Max.	+ 202	566	600	𝔄	248
Inf.	− 25	671	373	𝔅	227
Min.	− 249	774	149	ℭ	224
Inf.	+ 58	890	456	𝔇	307
Max.	+ 360	1001	758	𝔈	302
Inf.	− 18	1121	380	𝔉	378
Min.	− 398	1240	o	A	380
Inf.	− 42	1356	356	B	356
Max.	+ 326	1477	724	C	368
Inf.	+ 58	1584	456	D	268
Min.	− 219	1697	179	E	277
Inf.	o	1800	398	F	219
Max.	+ 219	1903	617	G	219
Inf.	− 58	2016	340	H	277
Min.	− 326	2123	72	I	268
Inf.	+ 42	2244	440	J	368
Max.	+ 398	2360	796	K	356
Inf.	+ 18	2479	416	L	380
Min.	− 360	2599	38	M	378
Inf.	− 58	2710	340	N	302
Max.	+ 249	2826	647	O	307
Inf.	+ 25	2929	423	P	224
Min.	− 202	3034	196	Q	227
Inf.	+ 46	3143	444	R	248
Max.	+ 287	3247	685	S	241
Inf.	− 56	3367	342	T	343
Min.	− 385	3480	13	U	329
Inf.	o	3600	398	V	385
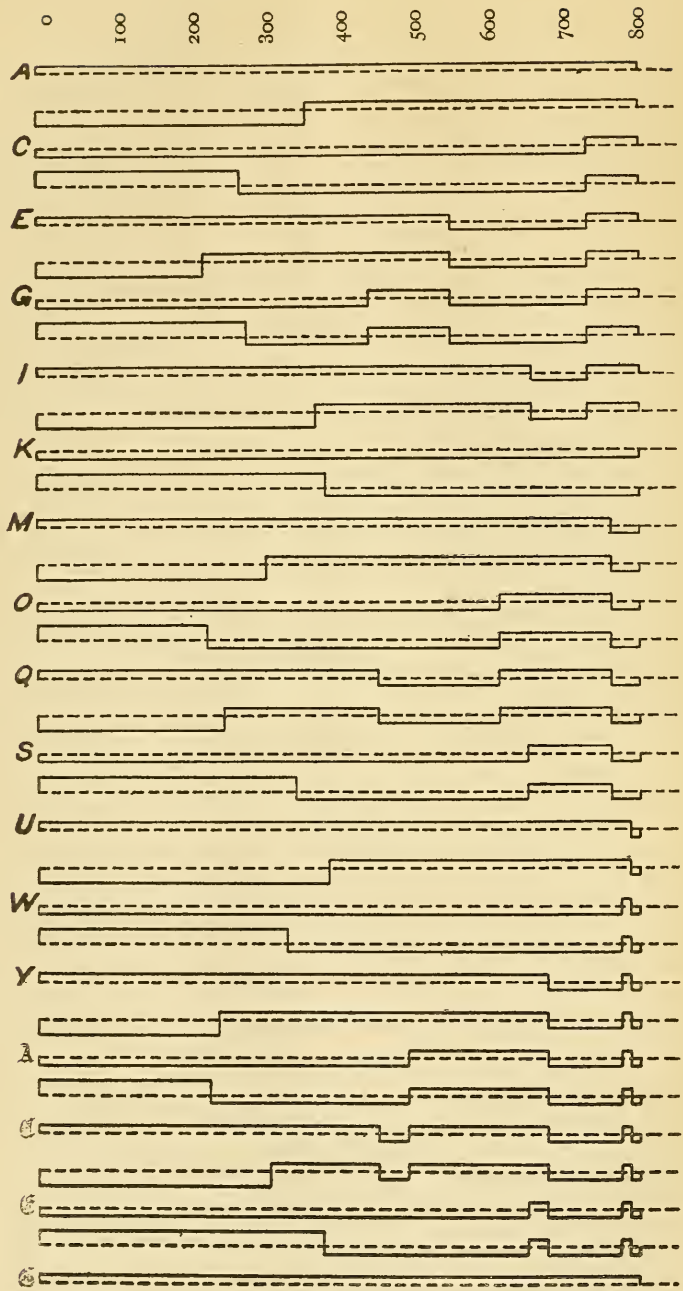
Fig. 39. The combination 5 and 8. Compare figure 38

C to E. Down from E to K, and up from to K to M. Down from M to O, and up from O to U. Down from U to W, and up from W to Y. Down from Y to 𝔄, and up from 𝔄 to 𝔊=A. The nerve ends located on these sections therefore receive five shocks during the period.

The five sections from the six hundred and forty-eighth to the six hundred and fifty-second move down from Y to C,

The tone 3
and up from C to I. Down from I to K, and up from K to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive three shocks during the period.

The sections of the partition from the six hundred and fifty-third to the seven hundred and seventieth move down

The tone 2
and up twice during the period. Let us examine those from the six hundred and fifty-third to the six hundred and seventy-second. They move down from Y to C, and up from C to M. Down from M to W, and up from W to Y. The nerve ends located on these sections therefore receive two shocks during the period.

The sections from the seven hundred and seventy-first to the seven hundred and eighty-third move down from U

The tone 1
to K, and up from K to U. The sections from the seven hundred and eighty-fourth to the seven hundred and ninety-sixth move down from A to K, and up from K to 𝔊=A. The nerve ends located on these sections therefore receive one shock during the period.

The relative intensities
The relative intensities of the several tones under the third, fourth, and sixth provisional assumptions are shown in the following table:

Tones	8 smooth	8 rough	5	3	2	1
Intensities	438	13	196	5	118	26
Percentages	55.0	1.6	24.6	.6	14.8	3.3

It is interesting to compare the intensities of the several tones in the last five cases, all representing the combination 8 plus

Comparison of the last five cases

5 of stirrup movement, but differing in the relative amplitudes of 8 and 5. The table contains the percentages of the five preceding tables. The first two columns show the ratio of the amplitudes of the stirrup movements of 8 and 5 For example, in the first case this ratio is as 3 : 1 or seventy-five to twenty-five; in the fifth case as 1 : 3 or twenty-five to seventy-five. The columns to the right contain the relative intensities of the several tones calculated under the provisional assumptions.

Amplitudes of stirrup movement		Subjective (theoretic) intensity					
8	5	8	6	5	3	2	1
75	25	56.6	——	24.6	.6	14.8	3.3
67	33	42.7	——	33.2	1.0	19.1	4.0
50	50	16.1	3.0	48.0	1.8	25.6	5.5
33	67	13.4	9.7	42.5	1.5	27.2	5.7
25	75	——	——	69.9	.9	23.9	5.3

We notice that the tone 8 decreases in intensity from 56.6 to 42.7, to 16.1, to 13.4, and finally disappears entirely. This latter case, however, does not mean that now the tone 5 is

alone audible. We see from the table that even now, in addition to 5, the very weak difference tone 3 and the fairly strong difference tones 2 and 1 are to be expected by the observer.[1]

As to the several difference tones, the most favorable condition for 6 seems to be, to have the component 5 of the compound stirrup movement somewhat more pronounced than 8. It appears, however, that in no case will this difference tone become very conspicuous. The most favorable condition for the difference tone 3 seems to be, to have the component 8 of stirrup movement about as strong as 5. The difference tones 2 and 1, on the other hand, appear with a maximum of intensity when the component 5 of stirrup movement is somewhat greater than 8. But their intensities are but little less in case the amplitudes of the two stirrup movements 8 and 5 are equal. With respect to all the difference tones taken together, it appears that these tones are very unfavorably influenced by a considerable difference in the amplitudes of the component stirrup movements, for no difference tone has a maximum intensity in either the first or the fifth case. And

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[1]Although this booklet is devoted to theory and not to experimental methods of research, I cannot refrain from mentioning a way of testing the theoretical results just spoken of, because it is so easy for any one who possesses a skillful hand and a trained ear, and the observation to be made is so pretty. No instruments are required but two good tuning forks on resonance boxes, accurately tuned in the ratio of 5:8, and a bass bow. The fork 5 must be sounded first, as strongly as possible, and it is necessary to have a fork which continues to sound strongly for quite a while. Then the bow is applied with the most delicate touch to the fork 8. It is necessary for the success of the experiment that the intensity of the higher tone vibration be increased from zero very slowly and uniformly. If these conditions are fulfilled, one suddenly hears the low difference tones 1 and 2 being added distinctly to the tone 5, whereas of 8 no trace is yet audible. If now the fork 8 is left to itself, and the fork 5 is stopped by firmly touching it with a finger, the tone 5 together with the difference tones disappears, but immediately one hears with surprising clearness the tone 8, which a moment ago was entirely inaudible. No similar observation can be made with a strongly sounding fork 8 and a weakly sounding fork 5. According to our theoretic deduction the lower tone does not become inaudible when the amplitude of 8 is three times that of 5, but still has a respectable intensity.

a prevailing intensity of 8 seems to be even less favorable to the difference tones than a prevailing intensity of 5. All these conclusions have, of course, only a relative value, since taking into account the various provisional assumptions changes the result considerably.

Let us study one more combination of sinusoidal stirrup movements. We have had only one interval greater than an octave, the combination 4 and 9. But we did not, then, take into account the inflection points of the curve. Let us do this with the combination 3 and 8, taking the amplitude of 3 twice as great as that of 8. This ratio of the amplitudes is arbitrarily chosen. But the selection of equal amplitudes would be no less arbitrary. The curve in figure 40 represents the function

The combination 3 and 8. The amplitude of 3 twice that of 8

$$f(x) = 2\sin 3x + \sin 8x.$$

The table below contains the abscissa and ordinate values of the maxima, minima, and inflection points of the curve.
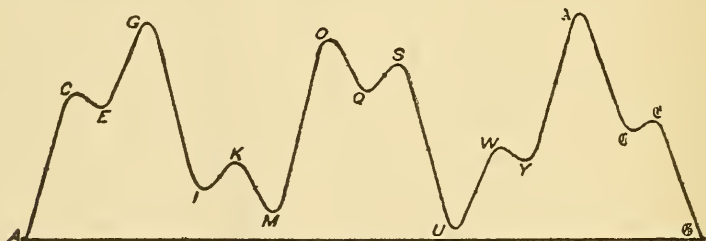
Fig. 40.   The combination 3 and 8

The successive positions of the partition corresponding to the maxima, minima, and inflection points are shown in figure 41.

The thirteen initial sections of the partition move down from $\mathfrak{F}$ to B, and up from B to D. Down from D to F, and

INTERVAL 3:8, AMPLITUDES 2:1

	Ordinate	Abscissa	Ordinate		Ordinate Difference
Inf.	0	0	297	N	228
Max.	+ 228	152	525	O	228
Inf.	+ 164	245	461	P	64
Min.	+ 95	353	392	Q	69
Inf.	+ 131	435	428	R	36
Max.	+ 165	510	462	S	34
Inf.	— 60	668	237	T	225
Min.	— 273	812	24	U	213
Inf.	— 171	920	126	V	102
Max.	— 55	1054	242	W	116
Inf.	— 73	1117	224	X	18
Min.	— 90	1177	207	Y	17
Inf.	+ 113	1337	410	Z	203
Max.	+ 297	1471	594	𝔄	184
Inf.	+ 152	1593	449	𝔅	145
Min.	— 13	1745	284	ℭ	165
Inf.	0	1800	297	𝔇	13
Max.	+ 13	1855	310	𝔈	13
Inf.	— 152	2007	145	𝔉	165
Min.	— 297	2129	0	A	145
Inf.	— 113	2263	184	B	184
Max.	+ 90	2423	387	C	203
Inf.	+ 73	2483	370	D	17
Min.	+ 55	2546	352	E	18
Inf.	+ 171	2680	468	F	116
Max.	+ 273	2788	570	G	102
Inf.	+ 60	2932	357	H	213
Min.	— 165	3090	132	I	225
Inf.	— 131	3165	166	J	34
Max.	— 95	3247	202	K	36
Inf.	— 164	3355	133	L	69
Min.	— 228	3448	69	M	64
Inf.	0	3600	297	N	228

up from F to H.   Down from H to J, and up from J to L.
Down from L to N, and up from N to P.

The tone 8   Down from P to R, and up from R to T.
Down from T to V, and up from V
to X.   Down from X to Z, and up from Z to 𝔅.   Down from
𝔅 to 𝔇, and up from 𝔇 to 𝔉.   The nerve ends located on
these sections therefore receive eight shocks during the pe-
riod.

Let us examine the sections from the sixty-fifth to the
sixty-ninth. They move down from 𝔉 to B, and up from
B to E. Down from E to F, and up from F to H. Down from
H to K, and up from K to L. Down from L to N, and up from
N to Q. Down from Q to S, and up from S to T. Down from
T to V, and up from V to Y. Down fom Y to Z, and up from
Z to 𝔅. Down from 𝔅 to 𝔈, and up from 𝔈 to 𝔉.   The nerve
ends located on these sections therefore receive eight shocks
during the period.

The seventieth section moves down from 𝔉 to B, and up
from B to E. Down from E to F, and up from F to H. Down
from H to K, and up from K to M. Down from M to N, and
up from N to Q. Down from Q to S, and up from S to T.
Down from T to V, and up from V to Y. Down from Y to Z,
and up from Z to 𝔅. Down from 𝔅 to 𝔈, and up from 𝔈 to 𝔉.
The nerve ends located on this section therefore receive eight
shocks during the period.

The sections of the partition from the seventy-first to the
one hundred and second move down from 𝔉 to B, and up from

                B to E. Down from E to F, and up from
The tone 6   F to H. Down from H to N, and up from
                N to T. Down from T to V, and up
from V to Y. Down from Y to Z, and up from Z to 𝔅. Down
from 𝔅 to 𝔈, and up from 𝔈 to 𝔉.   The nerve ends located
on these sections therefore receive six shocks during the period.

The sections from the one hundred and third to the one
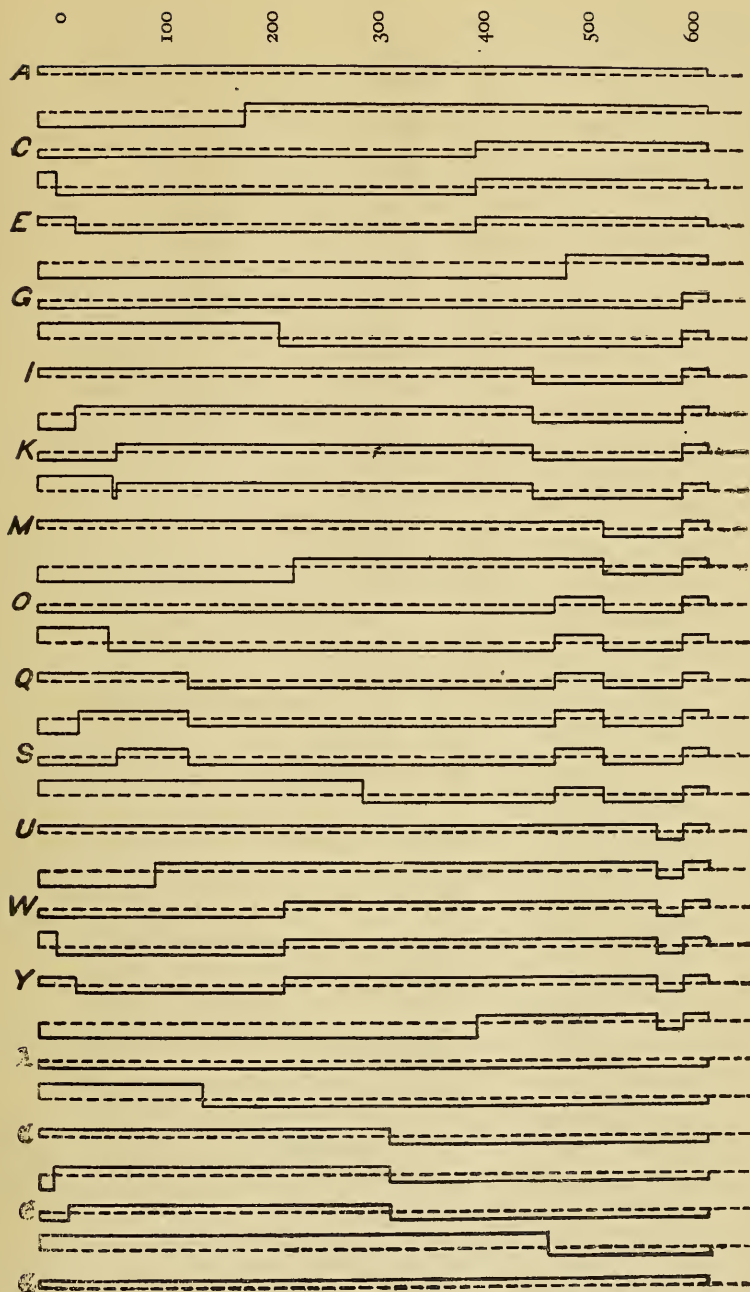
Fig. 41. The combination 3 and 8. Compare figure 40

hundred and forty-fifth move down from 𝔉 to B, and up from

The tone 5

B to E. Down from E to F, and up from
F to H. Down from H to N, and up from
N to T. Down from T to Z, and up from
Z to 𝔅. Down from 𝔅 to ℭ, and up from ℭ to 𝔉. The nerve
ends located on these sections therefore receive five shocks
during the period.

The sections from the one hundred and forty-sixth to the
one hundred and eighty-fourth move down from 𝔉 to B, and

The tone 4

up from B to E. Down from E to F, and
up from F to H. Down from H to N, and
up from N to T. Down from T to Z, and
up from Z to 𝔉. The nerve ends located on these sections
therefore receive four shocks during the period.

The sections from the one hundred and eighty-fifth to the
four hundred and fifty-sixth move down and up three times dur-

The tone 3

ing the period. Let us examine those from
the one hundred and eighty-fifth to the
two hundred and thirteenth. They move
down from 𝔉 to F, and up from F to H. Down from H to N,
and up from N to T. Down from T to Z, and up from Z to
𝔉. The nerve ends located on these sections therefore receive
three shocks during the period.

The sections from the four hundred and fifty-seventh to
the four hundred and sixty-eighth move down from A to F, and

The tone 2

up from F to M. Down from M to 𝔄, and
up from 𝔄 to 𝔊=A. The sections from
the four hundred and sixty-ninth to the
five hundred and first move down from A to G, and up from
G to M. Down from M to 𝔄, and up from 𝔄 to 𝔊=A. The
sections from the five hundred and second to the five hundred
and forty-sixth move down from A to G, and up from G to U.
Down from U to 𝔄 and up from 𝔄 to 𝔊=A.   The nerve ends
located on these sections therefore receive two shocks during
the period.

The sections of the partition from the five hundred and forty-seventh to the five hundred and seventieth move down from A to G, and up from G to $\mathfrak{G}=A$. The sections from the five hundred and seventy-first to the five hundred and ninety-fourth move down from A to $\mathfrak{A}$, and up from $\mathfrak{A}$ to $\mathfrak{G}=A$. The nerve ends located on these sections therefore receive one shock during the period.

The tone 1

The relative intensities of the several tones under the third, fourth, and sixth provisional assumptions are shown in the following table:

The relative intensities

Tones	8	6	5	4	3	2	1
Intensities	70	32	43	39	272	90	48
Percentages	11.8	5.4	7.2	6.6	45.8	15.1	8.1

We notice that the tone 3 is theoretically by far the strongest, as is to be expected. Of the difference tones, the tones 2, 1, and 5 appear to be somewhat more pronounced than 4 and 6. Under different assumptions concerning the physical properties of the partition these results would, of course, be somewhat different.

Throughout our previous discussions we have never taken into account the possibility of the tone intensities being further modified by a more central nervous condition like the one usually referred to as Weber's law. All our various approximations towards the intensities of the nervous processes take into consideration only conditions in the peripheral organ. Whether the intensities thus found are modified more centrally in accordance with Weber's law or

Weber's law in audition

not, is a question which at present must be left entirely open, like so many others, because of lack of experimental data.

Whenever we have spoken of "amplitudes" we have meant exclusively the amplitudes of stirrup movement. In order to make use of our theory in experi-

Sounding bodies and stirrup movement mental investigations we must remember the fact that the stirrup movements result from movements of the tympanum, transmitted by a rather complicated system of levers, the auditory ossicles. It is quite probable that the vibratory movements of the stirrup—even when these movements are highly complex—are approximately like those received by the hammer, the ossicle attached to the tympanum. But no one knows as yet how close or remote this approximation is. We certainly have no right to regard this approximation as infinitely close, save by way of a provisional assumption.

The movements of the tympanum result from rhythmical changes of the density of the external air. These density changes, in experimental investigations, are sometimes produced by the vibrations of gaseous bodies, as in labial organ pipes; more frequently, however, by the vibrations of solid bodies, particularly of tuning forks on resonance boxes. Now, we must not think that by graphically recording—which is a comparatively easy method—the vibrations of a tuning fork, we obtain a record of the exact form of the resulting air waves. It has been experimentally and mathematically proved that the form of the resulting air waves must be more or less different from the form of the vibratory movement of the fork or other solid body. The cause of this alteration of the form is to be found in the fact that the layer of air which adjoins the solid body and therefore directly receives the impulses from that body, is unsymmetric with respect to its elastic properties, because

it is in contact on one side with a practically unyielding body, on the opposite side with the easily yielding air.

It is of the utmost importance, therefore, if we wish to develop the theory by experimental investigation, to keep free from the delusion that any of the above theoretic results, say, in the case of the combination 5 and 8 with equal amplitudes, applies to what we hear in case two tuning forks of the vibration ratio 5 : 8, standing at an arbitrary distance from our ears and from the reflecting walls of our laboratory, vibrate with equal amplitudes. It is only by way of approximation that we can derive any theoretic conclusion from such an experiment. The starting point of our theory is the form of movement of the stirrup, not of external sounding bodies.

Under ordinary conditions, it is a great advantage that we possess two organs of hearing, some distance apart.   In experimental investigations, however, for the

The duality of our auditory organ

development of a theory of audition, this fact is often a serious obstacle. Since we cannot make experiments on audition while soaring like an eagle, any source of sound is likely to surround our body with standing waves, resulting from reflection. Let us regard the velocity of sound as three hundred and thirty meters, the distance between our ears as about fifteen centimeters. A tone of five hundred and fifty complete vibrations, that is, a tone representing the ordinary human voice quite well, has therefore a wave length of about sixty centimeters. The distance between a nodal point, where the rhythmic density changes of the air occur with full intensity, and a point of maximum vibratory movement, where there are practically no density changes affecting the tympanum, is then about fifteen centimeters. That is, it might happen with standing waves—if the head was kept perfectly still—that the amplitude of one of the components of stirrup movement would be almost zero in one ear, but very large in

the other, and every movement of the head would greatly alter these conditions; while the resulting consciousness would be, of course, the sum total of the tones heard by each ear. It is unnecessary to point out in further detail how this fact of hearing with two ears complicates the comparison of experimental results with the theoretical deductions of the present study, which refer only to one stirrup and one inner ear, and to an unalterable form of the components of stirrup movement in a given case.

The fact that we have two ears would be irrelevant only with exceedingly high tones, whose wave lengths in air would be so small as to be negligible quantities in comparison with the distance between our ears, as the wave lengths of light are negligible quantities in comparison with the distance between our eyes and even with the sensory elements of each eye.

Every one is familiar with the comparative clearness with which the ticking of a watch or the sound of a tuning fork is perceived if the vibrating body is firmly **Hearing without** pressed on the head or against the teeth. **the ear drum** Some believe that the physiological function of the ear in such a case is not essentially different from hearing under ordinary conditions; that the sound waves, the rhythmic changes of molecular density, which pass through the head, naturally pass also through the cavities of the head, of which one, the middle ear, particularly concerns us here.   As soon as rhythmic changes of density occur in the air of the middle ear, the tympanum adjusts itself to them by rhythmically moving back and forth. The stirrup cannot help following the tympanum, and so on. The only difference between this case and a case of ordinary hearing consists in the fact that the changes of density of the air affecting the tympanum originate on the inside of the tympanum in-

stead of on the outside, and that they must, on the whole, be much weaker in the former case.

There can be little doubt that the process just spoken of actually occurs. Some have insisted also on the possibility of hearing when the middle ear is destroyed and no movements of the stirrup occur. There is no reason why we should a priori deny the possibility of a shock being received by the nerve ends whenever a rhythmical change of molecular density takes its path directly through them. Such a molecular wave might originate from a vibrating solid body being pressed against skull or teeth, or from sound waves in the air striking the head and passing through it.

We must not overlook the fact, however, that even when the tympanum is totally destroyed, if sounds are perceived, the perception need not be the result of the sound waves simply passing through the nerves. Even in such a case stirrup movements are not excluded. If we blow over the mouth of a bottle, we cause rhythmical changes of density within the bottle, and, as a natural consequence, the air in the neck of the bottle rushes back and forth. These movements may often be observed with the naked eye when a fiber adherent to the inside of the neck of a bottle is forced by friction to follow the movements of the air. Now, when rhythmic changes of density occur in a middle ear whose tympanum is destroyed, there must naturally occur a back and forth movement of the air in the air passage, just as in the neck of a bottle. These back and forth movements of the air may cause by friction corresponding movements of the hammer and anvil and thus of the stirrup. No doubt, stirrup movements which are caused in this way must be of small magnitude. But no one who knows the surprisingly small amount of mechanical energy which is sufficient to call forth a response of the auditory organ will deny that they might result in an auditory sensation.

If not only a part or the whole of the tympanum is destroyed, but the chain of ossicles is also lost, the mechanical processes in the inner ear could be brought about by pressure differences on the two windows. An air wave, coming in through the external passage and the open middle ear, would at any given moment affect the two windows with a slightly different phase, arriving at one window a little earlier than at the other. This difference of phase means, of course, a difference of air pressure on the windows, and a difference of air pressure on the windows, according to the laws of mechanics, results in a movement of the internal fluid from the point of higher to that of lower pressure. It is plain, however, that this difference of phase, owing to the small distance between the two windows, must be very slight; and hearing which results in this way must be rather weak. But its possibility cannot be doubted.

Few cases, therefore, will be found where a sound is heard and we have to have recourse to the rather improbable assumption that the mere passing of molecular waves of density changes through the head and, thus, through the auditory nerve ends directly results in some weak response of the nerves. Nevertheless at least we may admit this assumption as possible. To admit it as possible would not cause any difficulty in comprehending the ordinary phenomena of audition, which might thus seem to become more complicated because such density waves must, of course, pass through the head whenever anybody hears anything. But such effects on the nerve ends, granted that they always exist, must ordinarily be overpowered by the incomparably stronger stimulations simultaneously received by the nerve ends by way of the stirrup movement.

Having studied the function of the human ear, it is in-

teresting to compare this with the organ of hearing of the lower vertebrates. Figure 42 indicates the manner of evolution of the cochlea. An original pit (Fig. 42 a) as found in a frog is gradually lengthened and assumes in the birds a banana-like shape (Fig. 42 b), showing a distinct tendency to coil. In mammals the process of lengthening and coiling has proceeded so far that the organ (Fig. 42 c), if it were transparent, would appear as a spiral. It is clear that the coiling can have little influence on the mechanical function of the organ. The lengthening of the organ, however, is of the utmost functional importance. The original pit does not differ materially from the other cavities which we find within

Comparative anatomy of the auditory organ
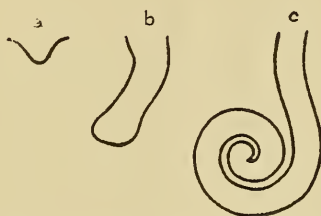
Fig. 42.   Evolution of the auditory organ

the labyrinth, communicating with the semicircular canals. In this pit movements of the fluid caused by movements of the stirrup—or rather columella plate, since the lower vertebrates have a much simpler connection of tympanum and oval window—produce, probably by mere friction, stimulation of the endings of the auditory nerve. The organ of the birds must function more nearly like the human organ, excepting the difference of function resulting from the fact that the endings of the auditory nerve are spread out over a small linear extent, whereas in the mammals they are distributed over a long distance.

In birds one can hardly speak of some nerve ends being farther away from the windows than others.

It is of some interest, in this connection, to note that animals with a short tube, as the birds, do not possess in the partition of the tube the pillars of Corti. They can get along without these pillars. And naturally. The longer the tube, the greater is the maximum pressure which may act upon the partition near the windows, in case the bulging of the partition is forced to proceed far towards the end of the tube. The greater the possible pressure, the greater is, of course, the need of a skeleton-like support in order to protect the sensitive cells from collapsing. Thus the mammals need the pillars because of the greater length of the tube.

What must be the difference of sound perception resulting from these anatomical differences in various species of animals? We saw that the human ear can

Comparative psychology of the sense of hearing

perceive several tones at the same time because the linear extension of the auditory organ permits the compound mechanical processes, transmitted from the stirrup to the fluid of the cochlea, to be analyzed into much simpler mechanical processes taking place in successive sections of the partition. It is plain, then, that in the auditory pit of a frog no analysis is possible. The result must be that the frog's ear can perceive only one tone at any moment; and this tone is most probably, as a rule, the highest of the several tones heard simultaneously under the same circumstances by the human ear.

The bird's ear, as we have seen, is intermediate between the frog's ear and the human ear. But it does not seem very probable that even birds can perceive very many tones simultaneously. The fact that birds sing is no indication to the contrary, since their song does not consist—like orchestral music —of simultaneous, but only of successive tones. Of more significance, in this respect, is the fact that some birds, for ex-

ample, parrots, are able to imitate human speech sounds. Speech sounds are characterized, according to the present state of phonetics, by particular groupings of tones in both simultaneity and succession. It is not certain that the rough imitation of human speech sounds by parrots is more than an imitation of the successive groupings of tones. Granted even that the birds possess the ability to perceive more than one tone simultaneously, the anatomical facts would make it probable that this ability is very limited in comparison with the human ear which perceives the most varied combinations of tones in speech sounds and in harmonic music.

Let us now briefly look back upon what we have done. We have regarded the organ of hearing as a long and narrow tube, filled with a practically incompressible fluid and divided lengthwise by an imperfectly elastic partition which is the seat of the auditory nerve ends. We have found that the problem of determining exactly, for each given form of stirrup movement, the mechanical processes taking place in the tube is from the mathematical side an almost hopelessly complex one, made still more difficult by the lack of data concerning the mere facts of hearing as well as the elastic and other physical properties of the partition. In order to overcome the intrinsic and accidental difficulties standing in our way, we have introduced six simplifying provisional assumptions; not using all six in every case, but now some of them, now others, according as the purpose of the moment seems to warrant. We have thus obtained a superficial, but for a beginning satisfactory, insight into the wonderful machinery by which we analyze the complicated sound waves with a result which—for example, with respect to the hearing of difference tones—is most surprising to one who knows nothing of the mechanics of the inner ear.

The need of experimental data

The theory thus developed does not pretend to be the ultimate solution of the problems attacked. We do not possess the data upon which to found a final theory. But we shall scarcely obtain these **The necessity** data without the guidance of a the**of a** ory. Experimental research must be **theory** systematic, must start from a theory, however imperfect this may be, in order to lead to scientific advancement. If the theory here offered succeeds in stimulating experimental research in a field somewhat neglected for many years, the author's hope will be realized.

APPENDIX

A list of former publications by the same author concerning the mechanics of the inner ear:

Über Kombinationstöne und einige hierzu in Beziehung stehende akustische Erscheinungen. Zeitschrift für Psychologie und Physiologie der Sinnesorgane **11**, 177-229. 1896.

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Karl L. Schäfer's "Neue Erklärung der subjectiven Combinationstöne." Ibid. **81**, 49-60. 1900.

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Zur Theorie der Geräuschempfindungen. Zeitschrift für Psychologie und Physiologie der Sinnesorgane **31**, 233-247. 1903.

Über Kombinations-und Asymmetrietöne. Annalen der Physik (Vierte Folge) **12**, 889-892. 1903.

The significance of wave-form for our comprehension of audition. American journal of psychology **18**, 170-176. 1907.

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